QUANTUM COHOMOLOGY OF A HILBERT SCHEME OF A HIRZEBRUCH SURFACE

BY

YONG FU

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

Doctoral Committee:

Professor Thomas Nevins, Chair
Professor Sheldon Katz
Professor Steven Bradlow
Associate Professor Hal Schenck

Abstract

In this thesis, we first use a ${\mathbb{C}^*}^2$-action on the Hilbert scheme of two points on a Hirzebruch surface to compute all one-pointed and some two-pointed Gromov-Witten invariants via virtual localization, then making intensive use of the associativity law satisfied by quantum product, calculate other Gromov-Witten invariants sufficient for us to determine the structure of quantum cohomology ring of the Hilbert scheme. The novel point of this work is that we manage to avoid families of invariant curves with the freedom of choosing cycles to apply the virtual localization method.

Acknowledgments

First, I would like to express my deep gratitude to my advisor Professor Sheldon Katz. During the long period of writing this dissertation, his encouragement and guidance are always the source of drive for me to move forward, without which this thesis would not have been possible. I have learned a lot of mathematics from talking and communicating with him, which has heavily shaped my way of thinking about mathematics. He keeps asking questions, which put my research in firm and solid footing and which also improve and strengthen many aspects of this work. I'm grateful to him for all what he has done for me.

I would like to thank Rahul Pandharipande and Tom Graber for helpful conversations. Their suggestions led me formulate Proposition 5.4.2 correctly.

I also want to take this opportunity to thank the members of my preliminary committee and dissertation committee. Their concern and encouragement have always pushed me to go ahead from the beginning of my research career. I also want to thank the teachers in UIUC who have taught me and broadened my perspective of understanding mathematics.

Finally, I sincerely thank my family members for their understanding and support in these years. Their endurance is the indispensable part for the completion of this work.

Table of Contents

Chapter 1

Introduction

In 1990's, enumerative geometry experienced a big impetus from the work by physicists, in which they calculated the number of rational curves of an arbitrary degree in a quintic threefold. Their work employed the idea of mirror symmetry in string theory to relate the counting of curves to a different problem in a mirror model in terms of physical arguments. To explain the phenomenon and verify their results, mathematicians made great efforts to establish a rigorous theory, the so-called Gromov-Witten theory. Now this subject is an established field and finds its applications in many areas of mathematics, for example, enumerative geometry, the theory of singularities, integrable systems to name just a few.

One of the task Gromov-Witten theory poses since its inception is how to compute Gromov-Witten invariants. Generally, this is a very hard problem. Some techniques have been developed to attack the problem, for example, degeneration method, finding mirror model to reduce the problem to period calculations, etc. When there is a torus action on the target space, computations can be relatively easily carried out with the so-called virtual localization [10], which is a modification of the usual topological localization.

Gromov-Witten invariants can be wrapped up as the quantum product on the cohomology ring of the space[7, 9]. This quantum product is associative and super-commutative, so that it makes the cohomology ring into a new ring called quantum cohomology ring. This new ring structure is a deformation of the cohomology ring of the space. If unraveled properly, the associativity of the quantum product exhibits very strong relations among Gromov-Witten invariants. These relations can be efficiently utilized for the computations of the invariants.

Since it was introduced, quantum cohomology has been computed for smooth toric varieties by Batyrev [1], for Grassmannian manifolds by Bertram, Daskalopoulos and Wenthworth[4], for Fano manifolds by Siebert and Tian[19], for flag varieties by Ciocan-Fontanine[5]. There are other cases where quantum cohomology is successfully decided. Recently, Okounkov and Pandharipande determined the ring structure of the equivariant quantum cohomology of the Hilbert scheme of points of a plane together with its relations to other areas of mathematics[15] and Pontoni established the quantum cohomology of the Hilbert scheme of two points of $P^1 \times P^1$ and studied its enumerative applications[16].

In this thesis, we take the example of a Hilbert scheme of two points over a Hirzebruch surface to compute its quantum cohomology ring. This work shows how the method of virtual localization and the associativity of quantum product are effectively combined to produce new results. Since Hirzebruch surfaces admit a $\mathbb{C}^{*2}$-action, the Hilbert scheme inherits this torus action. With this torus action, we succeed in computing many Gromov-Witten invariants of the Hilbert scheme using the virtual localization formula. These invariants suffice to determine the quantum products of the generators of the cohomology ring. In turn, the associativity of the quantum products of these generators provide us equations for the remaining Gromov-Witten invariants. Solving these equations, we obtain all two-pointed invariants. With these, the quantum ring structure is decided.

The layout of this thesis is as follows. In the second chapter, we first introduce the necessary background materials for Gromov-Witten theory with emphasis on the construction of virtual fundamental classes, then present the computational technique in terms of virtual localization. In the third chapter, we give an overview of the construction of Hilbert schemes on a smooth variety. In particular, Hilbert schemes of points on a smooth surface are smooth. We study the cohomology of the Hilbert schemes of two points over Hirzebruch surfaces, analyze the isolated or one-dimensional connected components of the $\mathbb{C}^{*2}$-action on the Hilbert schemes and determine the degrees of these components to prepare for computations of Gromov-Witten invariants. In the fourth chapter, we embark on the task of calculating Gromov-Witten invariants of one- or two-points. One type of three-point invariants are also computed. We explain why this method of localization only succeeds partially. In the last chapter, we first compute the quantum products of generator elements of the cohomology ring then, making heavy use of the associativity of the quantum product, establish equations for Gromov-Witten invariants, whose solutions provide necessary information to determine the quantum ring structure.

Chapter 2

Gromov-Witten Invariants and Localization

Gromov-Witten theory originated in both symplectic geometry and algebraic geometry almost in parallel at the early stage. It concerns with counting of maps from Riemann surfaces or algebraic curves to symplectic manifolds or smooth algebraic varieties under incidence conditions. Like many other mathematical problems in mathematics, legitimate counting comprises two steps: the first is compactification of the moduli problem; the second is the study of intersection theory on the moduli space. For Gromov-Witten theory, the first step is done by introduction of stable maps proposed by Kontsevich. The second step requires more effort to solve. It is quite a common phenomenon that moduli spaces have higher dimensions than expected. In the early stage of the development of the theory, Gromov-Witten invariants were successfully established by Ruan and Tian[17, 18] for the class of symplectic manifolds called weakly monotone symplectic manifolds, which include Fano and Calabi-Yau manifolds, using the fundamental class of the moduli space of stable maps. At about the same time, for homogeneous varieties and convex varieties, moduli spaces of stable maps of genus 0 are also proved to support a well-behaved intersection theory[9], so the Gromov-Witten theory for these classes of varieties can be firmly constructed.

Using the fundamental class of the moduli space to construct Gromov-Witten theory beyond the restrictive class of spaces is problematic when the moduli space has higher dimension than expected since it does not vary continuously when the space deforms. To construct invariants which are independent of deformations of a target space, the usual practice in differential category is to perturb some parameters to achieve the transversality conditions. But this method applied to moduli space of stable maps is more complicated in case it is possible and any perturbation in smooth category will force us to leave the algebraic realm. In mid-1990's, several groups came up with the solution, which is the construction of the virtual fundamental class. It is of the expected dimension in the moduli space, playing the role of fundamental class and realizing the fundamental class of the moduli space in case various transversality conditions are satisfied. The idea of the different constructions is using the excessive intersection theory to construct the class of the expected dimension in the moduli space. In algebraic geometry, people make use of a natural deformation-obstruction theory of the moduli problem to slice out the class via Fulton-McPherson's cone construction[8]. This is the approach we take in this work.

2.1 Moduli Spaces of Stable Maps

First we introduce the main objects of study.

Definition 2.1.1. [7, 9] *Given a smooth variety X over complex numbers. A morphism f from a prestable curve C with n-marked points $\{x_1, \cdots, x_n\}$ to X is called stable if the following conditions are satisfied:*

(1) every irreducible component of genus 0 which is contracted to one point in X by f must contain at least three special points (which means marked or nodal points);

(2) every irreducible component of genus 1 which is contracted to one point in X by f must contain at least one special point.

When X is a point, this agrees with the canonical notion of stable curves. An automorphism of the stable map f is defined to be an automorphism $\phi : C \to C$ such that $f\phi = f$ and $\phi(x_i) = x_i$ for $i = 1, \cdots, n$. It is easy to prove that the morphism f is stable if and only if it admits only finitely many automorphisms.

Let $\beta \in H_2(X, \mathbb{Z})$ be a homological class of degree 2 and g, n be nonnegative integers. We define the fibred category of stable maps over the category of schemes as follows: for any scheme S, the objects of the fibred category over S, $\overline{\mathcal{M}}_{g,n}(X, \beta)(S)$, is the set of all the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ S, \arrow[u, bend left, "x_i"] \end{tikzcd}$$

where $\pi : C \to S$ is a flat family of prestable curves, $x_i, i = 1, \cdots, n$ are sections of it, and $\forall s \in S, f_s : (C_s, x_{1s}, \cdots, x_{ns}) \to X$ is a stable map of genus g with n-marked points in the class β;

for any two such diagrams, the morphisms between them are commutative diagrams

$$\begin{tikzcd} C_1 \arrow[r, "\phi"] \arrow[rr, bend left, "f_1"] \arrow[d, "\pi_1"] & C_2 \arrow[r, "f_2"] \arrow[d, "\pi_2"] & X \\ S_1 \arrow[u, bend left, "x_i^{(1)}"] \arrow[r, "\psi"] & S_2, \arrow[u, bend left, "x_i^{(2)}"] & \end{tikzcd}$$

where the square is a fibred product.

Because stable maps only admit a finite number of automorphisms, this fibred category is a Deligne-Mumford stack. We denote it as $\overline{\mathcal{M}}_{g,n}(X, \beta)$. It is called the moduli space of stable maps.

2.2 Virtual Fundamental Class

2.2.1 Cotangent Complex

The traditional deformation-obstruction theory assumes new appearance in the hands of Behrend and Fantechi[3]. To present that, we need first to introduce the cotangent complex associated to a morphism $f : X \to Y$ between two algebraic schemes or (Artin) stacks. The (relative) cotangent complex of f or X over Y is an object $L^{\bullet}_{X/Y}$ in $D^-_c(X)$, which means the derived category over X of complexes bounded above with coherent cohomology.

Cotangent complexes have very nice properties[3, 12]:

(1) $h^i(L^{\bullet}_{X/Y}) = 0$ for $i > 0$;

(2) $h^0(L^{\bullet}_{X/Y}) = \Omega_{X/Y}$;

(3) Morphisms $f : X \to Y$ and $g : Y \to Z$ induce a distinguished triangle $f^* L^{\bullet}_{Y/Z} \to L^{\bullet}_{X/Z} \to L^{\bullet}_{X/Y}$ in $D^-_c(X)$;

(4) Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "j'"] & Y \end{tikzcd}$$

be a commutative square. Then there is a natural morphism

$$j'^* L^{\bullet}_{X/Y} \to L^{\bullet}_{X'/Y'}$$

obtained by composing the morphisms

$$j'^* L^\bullet_{X/Y} \to L^\bullet_{X'/Y} and L^\bullet_{X'/Y} \to L^\bullet_{X'/Y'}.$$

If the square is Cartesian, then

$$h^i(j'^* L^\bullet_{X/Y}) \to h^i(L^\bullet_{X'/Y'})$$

is an isomorphism for $i = 0$ and a surjection for $i = -1$.

(5) If $f : X \to X'$ is a closed embedding with the ideal sheaf $\mathcal{I}$ and $X' \to Y$ is a smooth morphism, then $L^\bullet_{X/Y \geq -1}$ is isomorphic to $[\mathcal{I}/\mathcal{I}^2 \to f^*\Omega_{X'/Y}]$, which are terms at -1 and 0.

Let X be a Deligne-Mumford stack and $T \to \overline{T}$ be a square-zero closed embedding with the ideal sheaf J. A morphism $g : T \to X$ induces canonical homomorphisms by property (3) above:

$$g^* L^\bullet_X \to L^\bullet_T \to L^\bullet_{T/\overline{T}}.$$

Since $L^\bullet_{T/\overline{T} \geq -1} = J[1]$ by (5), this homomorphism may be considered as an element $\omega(g) \in Ext^1(g^* L^\bullet_X, J)$. This class has the following properties:

- the morphism $g : T \to X$ extends to a morphism $g : \overline{T} \to X$ if and only if $\omega(g) = 0$;
- when $\omega(g) = 0$, the set of such extensions forms a torsor under $Ext^0(g^* L^\bullet_X, J)$.

2.2.2 Perfect Obstruction Theory

Definition 2.2.1. [3] *A perfect obstruction theory for X is a morphism $\phi : E^\bullet \to L^\bullet_X$ in $D^\bullet_c(X)$ from a two-term complex of vector bundles $E^\bullet = [E^{-1} \to E^0]$ to the cotangent complex $L^\bullet_X$ of X such that*

(1) $h^0(\phi)$ is an isomorphism,

(2) $h^{-1}(\phi)$ is a surjection.

$E^\bullet$ dualizes to $E_0 \to E_1$, which gives rise to the quotient stack $[E_1/E_0]$. It is proved that for a morphism $\phi : E^\bullet \to L^\bullet_X$, the following statements are equivalent[3]:

(i) $\phi : E^\bullet \to L^\bullet_X$ is a perfect obstruction theory;

(ii) $\phi^\vee : \mathcal{N}_X \to [E_1/E_0]$ is a closed embedding, where $\mathcal{N}_X$ is the intrinsic normal sheaf of X;

(iii) If $T \to \overline{T}$ is a square zero extension of schemes with the ideal sheaf J, and $g : T \to X$ is a morphism, then $\phi^*\omega(g) \in Ext^1(g^*E^\bullet, J)$ vanishes if and only if g can be extended to $\overline{T}$; and if $\phi^*\omega(g) = 0$, then all the extensions form a torsor under $Ext^0 g^*E^\bullet, J) = Hom(g^*h^0(E^\bullet), J)$.

Notice this last item (iii) is the bridge to the traditional deformation theory. Let $C(E^\bullet)$ be the fibred product in the following Cartesian diagram:

$$\begin{tikzcd} C(E^\bullet) \arrow[r] \arrow[d] & E_1 \arrow[d] \\ \mathcal{C}_X \arrow[r] & {[E_1/E_0],} \end{tikzcd}$$

where $\mathcal{C}_X$ is the intrinsic normal cone of X, a closed substack in $\mathcal{N}_X$. $C(E^\bullet)$ is a closed cone in the vector bundle E_1. Under the Gysin map $0^* : A_*(E_1) \to A_*(X)$, the image of the cycle class defined by this cone is an element $[X, E^\bullet]$ in $A_{rkE^0 - rkE^{-1}}(X, \mathbb{Q})$. It is called the virtual fundamental class of the obstruction theory.

It is almost a verbatim generalization to construct the virtual fundamental class from a relative obstruction theory.

Definition 2.2.2. [3] *A perfect relative obstruction theory for $X \to Y$ between two algebraic stacks of relative Deligne-Mumford type is a morphism $\phi : E^\bullet \to L^\bullet_{X/Y}$ in $D^\bullet_c(X)$ from a two-term complex of vector bundles $E^\bullet = [E^{-1} \to E^0]$ to the relative cotangent complex $L^\bullet_{X/Y}$, such that*

(1) $h^0(\phi)$ is an isomorphism,

(2) $h^{-1}(\phi)$ is a surjection.

The fibred product of E_1 and the relative intrinsic normal cone $\mathcal{C}_{X/Y}$ of $X \to Y$ over the stack $[E_1/E_0]$ is a closed cone inside E_1, which gives rise to the desired class in $A_{rkE^0 - rkE^{-1}}(X, \mathbb{Q})$ when X is Deligne-Mumford. The reason to come up with the relative version of a perfect obstruction theory is that for some moduli problems the relative deformation and obstruction theory is easier and more natural to understand and has a neater formulation. Gromov-Witten theory is such an example.

Recall for fixed schemes C and X, the first-order deformations of a morphism $f : C \to X$ are identified as $H^0(C, f^*TX)$, and the obstruction to deformations of f lives in $H^1(C, f^*TX)$. In the moduli space of stable maps, the domain curve C also varies, thus making the deformation and obstruction spaces $T^i, i = 1, 2$, very complicated to express. In fact, there is an exact sequence relating the various deformation and obstruction theories [3, 10]:

$$0 \to Ext^0(\Omega_C(D), \mathcal{O}_C) \to H^0(C, f^*TX) \to T^1 \to$$
$$\to Ext^1(\Omega_C(D), \mathcal{O}_C) \to H^1(C, f^*TX) \to T^2 \to 0,$$

in which D is the divisor of marked points on the domain curve C. A natural relative obstruction theory can be constructed as follows[2]. Let $\mathcal{M}_{g,n}$ and $\mathcal{C}$ denote the Artin stack and the universal family of prestable curves with arithmetic genus g and n-marked points and let $Mor(\mathcal{C}, X)$ be the stack of morphisms from $\mathcal{C}$ to X. Then as a family version of the above picture, there exists a natural perfect relative obstruction theory

$$\phi : R\pi_*(F^*T_X)^\vee \longrightarrow L^\bullet_{Mor(\mathcal{C},X)/\mathcal{M}_{g,n}},$$

where $F : \mathcal{C} \times_{\mathcal{M}_{g,n}} Mor(\mathcal{C}, X) \to X$, is the universal morphism and π is the projection from the product to the second factor. It can be proved that the moduli stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of stable maps is an open substack of $Mor(\mathcal{C}, X)$, so it inherits a relative obstruction theory by restriction. The virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ defined by this obstruction theory is the class to build Gromov-Witten invariants. By Riemann-Roch theorem, its degree is

$$\begin{aligned} & \chi(f^*TX) + dim\mathcal{M}_{g,n} \\ = \;& dimH^0(C, f^*TX) - dimH^1(C, f^*TX) + dim\mathcal{M}_{g,n} \\ = \;& c_1(X)\beta + (1-g)dimX + 3g - 3 + n. \end{aligned}$$

This number is denoted as $virdim\overline{\mathcal{M}}_{g,n}(X, \beta)$, called the virtual dimension of the moduli space. In reality the virtual fundamental class is built via a global resolution of the obstruction theory on $\overline{\mathcal{M}}_{g,n}(X, \beta)$, which means a two-term complex of vector bundles $[E^{-1} \to E^0]$ isomorphic to the deformation-obstruction complex in the derived category.

2.3 Definition of Gromov-Witten Invariants

Once we have described the construction of the virtual fundamental classes of the moduli spaces of stable maps, Gromov-Witten invariants and the quantum product over $H^*(X,\mathbb{Q})$ can be defined as follows[9, 7]. Each marked point gives rise to an evaluation map $ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow X, i = 1,\cdots,n$.

Definition 2.3.1. *Gromov-Witten invariants are*

$$I_{g,n,\beta} < a_1,\cdots,a_n >= \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} {ev_1}^* a_1 \cup \cdots \cup {ev_n}^* a_n,$$

for $a_i \in H^*(X,\mathbb{Q}), i = 1,\cdots,n$.

When there is no danger of confusion, we also write $I_{g,n,\beta} < a_1,\cdots,a_n >$ as $< a_1,\cdots,a_n >_{g,n,\beta}$ or $< a_1,\cdots,a_n >_\beta$. Note that Gromov-Witten invariants are equal to zero unless

$$\sum_{i=1}^{n} deg(a_i) = virdim \overline{\mathcal{M}}_{g,n}(X,\beta).$$

The gravitational descendants can be defined analogously. Recall that $\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ can be identified with the universal curve over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. So, for any $1 \leq i \leq n$, we have sections $s_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n+1}(X,\beta)$ of π mapping the stable map $\{f : (C, x_1,\cdots,x_n) \to X\}$ to the point x_i on C. We now define the i-th cotangent line to be the line bundle

$$L_i := s_i^* \omega_\pi,$$

on $\overline{\mathcal{M}}_{g,n}(X,\beta)$, where ω_π denotes the relative dualizing sheaf of π.

Definition 2.3.2. *For any non-negative integers* $m_1,\cdots,m_n$, *gravitational descendants are the numbers*

$$< \tau_{m_1} a_1,\cdots,\tau_{m_n} a_n >= \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} {ev_1}^* a_1 \cdot c_1(L_1)^{m_1} \cup \cdots \cup {ev_n}^* a_n \cdot c_1(L_n)^{m_n}.$$

Also, the gravitational descendants are equal to zero unless

$$\sum_{i=1}^{n}(deg(a_i)+m_i) = virdim\overline{\mathcal{M}}_{g,n}(X,\beta).$$

Given a basis $T_0 = 1, T_1, \cdots, T_m$ for $H^*(X,\mathbb{Q})$ of homogeneous elements.

Definition 2.3.3. *The quantum product of two elements $a, b \in H^*(X,\mathbb{Q})$*

$$a * b = \sum_i \sum_{\beta \in H_2(X,\mathbb{Z})} I_{0,3,\beta}(a,b,T_i)q^\beta T^i,$$

where $T^0, \cdots, T^m$ form the dual basis of $T_0 = 1, T_1, \cdots, T_m$ in the sense that $\int_X T^i \cup T_j = \delta_{ij}$, and q^β are formal variables satisfying $q^{\beta_1}q^{\beta_2} = q^{\beta_1+\beta_2}$ for any $\beta_1, \beta_2 \in H_2(X,\mathbb{Z})$.

We impose the convention that the degree of q^β is $\int_\beta c_1(TX)$ and the degrees of elements in $H^*(X,\mathbb{Q})$ are inherited. Then it can be proved[9, 7] that with this product, $H^*(X,\mathbb{Q}) \otimes \mathbb{Q}[[q^\beta : \beta \in H_2(X,\mathbb{Z})]]$ is an associative and super-commutative ring, called the (small) quantum cohomology ring of X. The associativity wraps up strong relations among Gromov-Witten invariants.

2.4 Equivariant Cohomology and K-Theory

In this section, we collect some basic facts about equivariant topology and localization formulas which are useful for our purpose.

2.4.1 Equivariant Cohomology

For a Lie group G, we ask the question whether there exists a principal G-bundle $EG \to BG$, with EG contractible. Such a bundle is universal in the topological setting: if $E \to B$ is any principal G-bundle, then there is a map $B \to BG$, unique up to homotopy, such that E is isomorphic to the pullback of EG. It is called a classifying space of the group G. For any compact Lie group G, it is well-known that its classifying space exists.

When the Lie group G acts on a topological space X, the equivariant cohomology is defined to be $H^*_G(X) = H^*(X \times_G EG)$. Note that if G acts on X freely, then the projection $X \times EG \to X$ induces a map $X \times_G EG \to X/G$, with the contractible fibre EG. So in this case, $H^*_G(X) = H^*(X/G)$.

A map between topological spaces $X \to Y$ induces a map $X \times_G EG \to Y \times_G EG$, so we have a ring homomorphism $H_G^*(Y) \to H_G^*(X)$. This means $H_G^*(\cdot)$ is a contravariant functor from the category of topological spaces to the category of rings. Since $pt \times_G EG = BG, H_G^*(pt) = H^*(BG)$. This is regarded as the coefficient ring of the equivariant cohomology, denoted as H_G^*, namely $H_G^*(X)$ has a module structure over H_G^*, induced by the map $X \to pt$ and hence $X \times_G EG \to BG$. It is well-known that for a torus $T = (S^1)^{\times n}, BT = (\mathbb{C}P^\infty)^{\times n}$, so $H_T^* = \mathbb{Z}[t_1, \cdots, t_n]$.

When V is an equivariant vector bundle over X,i.e. a G-vector bundle, $V \times_G EG$ is a vector bundle over $X \times_G EG$ of the same rank. Any characteristic class c of $X \times_G EG$ is called an equivariant characteristic class of X, denoted $c^G(V)$. The candidates we have in mind here for the characteristic class c are e.g. Euler class and Chern classes when V is a complex vector bundle, etc. All the relations of characteristic classes for ordinary vector bundles carry over to equivariant characteristic classes.

The equivariant integral over X is a well-defined H_G^*-linear map:

$$\int_X : H_G^*(X) \to H_G^*.$$

When a torus T acts on a manifold X, it is a fact that every connected component F is a smooth submanifold. Its normal bundle is denoted by N_F. This admits a T-action from the action of T on X, which is trivial on the base. The celebrated Atiyah-Bott localization formula says:

$$\int_X u = \sum_F \int_F \frac{u|_F}{e^T(N_F)}$$

for any $u \in \mathcal{R}_T H_T^*(X)$, where $\mathcal{R}_T$ is the field of fractions of H_T^*. Note that when $T = (S^1)^{\times n}, \mathcal{R}_T = \mathbb{Q}(t_1, \cdots, t_n)$.

Equivariant Chow groups $A_G^*(X)$ can also be defined in nice situations in algebraic geometry when an algebraic group G acts on a projective variety or a Deligne-Mumford stack X and the localization formula for torus actions also holds. When $T = (\mathbb{C}^*)^{\times n}$, we still have $A_*^T(pt) = \mathbb{Z}[t_1, \cdots, t_n]$, so again its fraction field is $\mathcal{R}_T = \mathbb{Q}(t_1, \cdots, t_n)$.

Let's recall the relevant construction of topological Gysin maps in the usual and equivariant situations.

Suppose $f : Y \to X$ is a proper map between two smooth manifolds Y and X. The Gysin map is a homomorphism of cohomology groups

$$f_* : H^i(Y) \to H^{i+d}(X),$$

where $d = dimX - dimY$. It has the following properties:

(i)(Functoriality) For proper maps $g : Z \to Y$ and $f : Y \to X, fg$ is also proper and $(fg)_* = f_* g_*$.

(ii)(Projection formula) For $a \in H^*(X)$ and $b \in H^*(Y), f_*(f^*a \cdot b) = a \cdot f_*b$.

(iii)(Naturality) Given a fiber square of smooth manifolds and maps

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

with f and f' proper, and $dimY - dimX = dimY' - dimX'$, then

$$g^* f_* = (f')_*(g')^*.$$

(iv)(Embedding) If $f : Y \hookrightarrow X$ is a closed embedding of codimension d with the normal bundle N, then the composition $f^* f_* : H^i(Y) \to H^{i+d}(Y)$ is multiplication by the Euler class $e(N)$.

(v)If $g : Y' \to X$ is a map, and $g(Y') \cap f(Y) = \emptyset$, then $g^* f_* = 0$.

When a Lie group G acts on Y and X and f is an equivariant proper map from Y to X, there is an equivariant version of Gysin maps, still denoted as $f_* : H^i_G(Y) \to H^{i+d}_G(X)$. It satisfies the similar properties as listed above, but in the property(iv), $e(N)$ needs to be replaced by the equivariant Euler class $e^G(N)$ of the normal bundle. In particular, from property (iv), we have a very useful

Corollary 2.4.1. *If $f : Y \hookrightarrow X$ is a closed equivariant embedding of codimension d with normal bundle N and $i : p \hookrightarrow Y$ is a fixed point, then $i^* f_*(Y) = e^G(N|_p)$, which can be written as the product of weights of the restriction of N to p.*

2.4.2 Equivariant K-Theory

We also need to recall the definition of equivariant K-theory. Let G act on a compact space X. The set of isomorphism classes of G-vector bundles on X forms an abelian semigroup under direct sum. Its abelianization is called equivariant K-group, denoted as $K_G(X)$: its elements are formal differences $E_0 - E_1$ of G-vector bundles on X, modulo the equivalence relation $E_0 - E_1 = {E'}_0 - {E'}_1$ if and only if $E_0 \oplus {E'}_1 \oplus F \cong {E'}_0 \oplus E_1 \oplus F$ equivariantly for a G-bundle F on X.

The tensor product of G-vector bundles induces a structure of commutative ring in $K_G(X)$. Because a G-equivariant map between two G-spaces pulls back a G-vector bundle to a G-vector bundle, K_G is a contravariant functor from compact G-spaces to commutative rings. If $G = 1$, we write $K(X)$ for $K_G(X)$. This is the ordinary K-theory.

If X is a point, then $K_G(X)$ is denoted as $R(G)$, the representation ring or the character ring of G. When G is a torus T, $R(T) = \mathbb{Z}[t_1, \cdots, t_n]$. For any G-space X, $K_G(X)$ is a module over $R(G)$, induced from the map $X \to pt$.

Taking Chern character induces an isomorphism

$$ch : K(X) \otimes \mathbb{Q} \to H^*(X, \mathbb{Q})$$

between rings; also taking equivariant Chern character induces a homomorphism

$$ch^G : K_G(X) \otimes \mathbb{Q} \to H^*_G(X, \mathbb{Q})$$

between rings. But ch^G is not necessarily an isomorphism.

2.4.3 Virtual Localization

One of the few computation tools for Gromov-Witten invariants is the virtual localization technique when the variety X admits a group action.

Let's get back to the general situation, where a torus group T acts on X and on the perfect obstruction theory $E^\bullet$ equivariantly. Let $X_i, i \in \mathcal{I}$ be a connected component of the torus action on X. It is a fact that every coherent sheaf $\mathcal{F}$ on X_i decomposes into the direct sum of the fixed part $\mathcal{F}^f$ and the moving part $\mathcal{F}^m$ under the group action. It can be proved that the fixed part

$E_i^{\bullet,f}$ of the restriction of $E^\bullet$ to the component induces a perfect obstruction theory for X_i, which in turn produces the virtual fundamental class $[X_i, E_i^{\bullet,f}]$ or $[X_i]^{vir}$. Let $N_i^{vir} = E_0^m - E_1^m$ be the virtual bundle of the moving part of the restriction to X_i of $E_\bullet$, the dual to $E^\bullet$, called the virtual normal bundle. Then the virtual localization formula is[7]

$$\int_{[X]^{vir}} u = \sum_{i \in \mathcal{I}} \int_{[X_i]^{vir}} \frac{u|_{X_i}}{e^T(N_i^{vir})},$$

for any $u \in \mathcal{R}_T A_*^T(X)$, where $e^T(N_i^{vir}) = e^T(E_0^m - E_1^m) = \dfrac{e^T(E_0^m)}{e^T(E_1^m)}$ is the equivariant Euler class of N_i^{vir} on X_i. When $u \in A_*(X)$ has the same degree as $[X]^{vir}$ and invariant under the T-action, we take an equivariant lifting u^T of u in $A_*^T(X)$. Since

$$\int_{[X]^{vir}} u = \int_{[X_i]^{vir}} u^T \in \mathbb{Q},$$

this provides a way to compute the ordinary integrals over the virtual fundamental class via the equivariant integrals over the connected components of fixed points using localization formula.

When a torus T acts on a smooth projective variety X, the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ automatically inherits a torus action by composing with the action on X. In some cases, the connected components can be identified and recorded by graphs. With a resolution $E^\bullet$ of the canonical obstruction theory, the tangent space T^1 and the obstruction space T^2 are built into the exact sequence

$$0 \to T^1 \to E_0 \to E_1 \to T^2 \to 0,$$

and T^1 and T^2 are related to other terms as in the exact sequence in §2.2. This provides us a way to calculate the equivariant Euler class of the virtual normal bundle N_i^{vir} on each connected component of the fixed loci in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and hence Gromov-Witten invariants of X in terms of the virtual localization[10].

Chapter 3

Hilbert Schemes of Points

Hilbert schemes parametrize the subschemes in a given scheme with some fixed numerical data. They are rare and precious examples in which the moduli problems are finely represented. Because of this, they work as the ground for many other moduli problems. We are here interested in the Hilbert schemes of two points when the given scheme is an algebraic surface, especially a Hirzebruch surface. We want to make use of a torus action on the Hirzebruch surface to compute the Gromov-Witten invariants of the Hilbert scheme.

3.1 Definition of Hilbert Schemes

Let X be a projective scheme over $\mathbb{C}$ and $\mathcal{O}(1)$ be an ample line bundle over X. Hilbert schemes describe the moduli of the closed subschemes in X with some numerical invariants.

Given a polynomial $P(n)$, we consider the contravariant functor

$$\mathcal{H}ilb^P : (schemes/\mathbb{C}) \to (sets),$$

where, for any scheme S over $\mathbb{C}$,

$$\begin{aligned} \mathcal{H}ilb^P(S) = \{Y \subset X \times S \text{ a closed subscheme} : Y \to S \text{ is flat} \\ \text{and for any } s \in S, Y_s \text{ has Hilbert polynomial } P(n)\}. \end{aligned}$$

Here the Hilbert polynomial of Y_s is that of its structure sheaf. Let's recall that the Hilbert polynomial of a coherent sheaf $\mathcal{F}$ on X is defined by

$$P(n) = \chi(\mathcal{F} \otimes \mathcal{O}(n)),$$

for any non-negative integer n. Since $Y \to S$ is flat, the Hilbert polynomial of Y_s is independent of s. So it's reasonable to impose the condition that each fiber have the given Hilbert polynomial. A celebrated theorem of Grothendieck is

Theorem 3.1.1. *The functor $\mathcal{H}ilb^P$ is represented by a projective scheme $Hilb^P/\mathbb{C}$.*

This means that there exists a closed subscheme $U \subset X \times Hilb^P$, for which $U \to Hilb^P$ is flat and every fiber has the Hilbert polynomial $P(n)$, such that any family $Y \to S$ is induced from a unique morphism $S \to Hilb^P$.

Let P be the constant polynomial n. We denote by $X^{[n]}$ the corresponding Hilbert scheme and call it the Hilbert scheme of n-points. As the degree of the Hilbert polynomial is the dimension of the subschemes, $X^{[n]}$ parametrizes 0-dimensional subschemes of length n in X, i.e.

$$dim H^0(Z, \mathcal{O}_Z) = \sum_{p \in supp(Z)} dim(\mathcal{O}_{Z,p}) = n.$$

One type of points in this Hilbert scheme $X^{[n]}$ are all the pairwise distinct n-points in X. They form a dense open subset in $X^{[n]}$. Usually, Hilbert schemes are singular, non-reduced, reducible and even not connected. But when X is a smooth algebraic surface, a result of Fogarty says $X^{[n]}$ is also smooth and irreducible of dimension $2n$.

For a smooth surface X, its n-fold symmetric product $X^{(n)}$ is defined to be

$$\overbrace{X \times \cdots \times X}^{n} / S^n,$$

where S^n is the symmetric group acting as shuffling the tuples. It is an orbifold. There is a natural morphism

$$\pi : X^{[n]} \longrightarrow X^{(n)},$$

defined by

$$\pi(Z) = \sum_{p \in supp(Z)} dim(\mathcal{O}_{Z,p})p,$$

for any $Z \in X^{[n]}$. This map can be regarded as the resolution of singularities of $X^{(n)}$.

3.2 Cohomology of Hilbert Schemes of Two Points of Surfaces

It is a fact that when a torus acts on a smooth projective variety with finitely many fixed points, its Chow groups and homology groups of the same degrees agree and these groups don't admit any torsion. As we'll see this is the case for a Hirzebruch surface $F_a, a \geq 1$ and for $F_a^{[2]}$, the Hilbert scheme of two points on F_a. For this reason, when we study their cohomology groups, we can work out their homology groups by Poincare duality whenever this is more convenient. In order to determine the homology groups of the Hilbert scheme, let's recall the blowup construction of the Hilbert scheme of two points on a general algebraic surface S.

Let $\Delta : S \to S \times S$ be the diagonal morphism. The blowup $Bl_\Delta(S \times S)$ of $S \times S$ along the diagonal Δ, denoted as $\widetilde{S \times S}$, is a four-dimensional smooth variety with the exceptional divisor $P(TS)$, denoted as $\widetilde{S}$, the projectivization of the tangent bundle of S. $\mathbb{Z}_2$ acts on $S \times S$ by exchanging the order of points, which fixes the diagonal. It automatically induces an involution on the blowup, which fixes the exceptional divisor. This in turn induces an involution on the homology groups. The Hilbert scheme $S^{[2]}$ is the quotient of the blowup under the involution. We go back and forth between the two points of view for the Hilbert scheme when necessary.

Let's use $\phi : \widetilde{S \times S} \to S^{[2]}$ to denote this quotient. By Example 1.7.6 in [8], there is a canonical isomorphism

$$\phi^* : A_*(S^{[2]}) \to A_*(\widetilde{S \times S})^{\mathbb{Z}_2},$$

where for a subvariety V in $S^{[2]}$,

$$\phi^*[V] = \sum e_W[W],$$

the sum over all irreducible components of $\phi^{-1}(V)$, and $e_W = \#\{g \in \mathbb{Z}_2 : g|_W = id_W\}$, and where

$A_*(\widetilde{S \times S})^{\mathbb{Z}_2}$ is the fixed subgroup of $A_*(\widetilde{S \times S})$ under the involution. We have identities

$$\phi^*\phi_* = 2id, \quad \phi_*\phi^* = 2id.$$

The intersection product operations in these two rings are related in the following formula. For any $x, y \in A_*(S^{[2]})$, we have (see Example 8.3.12 in [8])

$$x \cdot y = \frac{1}{2}\phi_*(\phi^* x \cdot \phi^* y).$$

The above blowup construction can be shown in the fibre square

$$\begin{tikzcd} \widetilde{S} \arrow[r, "j"] \arrow[d, "g"'] & \widetilde{S \times S} \arrow[d, "f"] \\ S \arrow[r, "\Delta"] & S \times S, \end{tikzcd}$$

where j is the inclusion of the exceptional divisor, and f, g are the projections.

Let T be the tautological bundle $\mathcal{O}(-1)$ on $\widetilde{S}$, which is also the normal bundle of the exceptional divisor in $\widetilde{S \times S}$, and $E = g^*(TS)/T$ be the quotient bundle on $\widetilde{S}$, in other words, we have the exact sequence

$$0 \to T \to g^*(TS) \to E \to 0.$$

Because f is a l.c.i. morphism of relative dimension zero, the Gysin map $f^* : A_*(S \times S) \to A_*(\widetilde{S \times S})$ is well-defined. The following proposition is copied from Proposition 6.7 and Example 8.3.9 in [8].

Proposition 3.2.1. *There are split exact sequences*

$$0 \to A_k(S) \xrightarrow{\alpha} A_k(\widetilde{S}) \bigoplus A_k(S \times S) \xrightarrow{\beta} A_k(\widetilde{S \times S}) \to 0,$$

with $\alpha(x) = (c_1(E) \cap g^*x, -\Delta_* x)$ *and* $\beta(\widetilde{x}, y) = j_*\widetilde{x} + f^*y$. *A left inverse for* α *is given by* $(\widetilde{x}, y) \mapsto g_*(\widetilde{x})$. *The ring structure of* $A^*(\widetilde{S \times S})$ *is determined by the following rules:*

(i) $f^*y \cdot f^*y' = f^*(y \cdot y')$;

(ii) $j_*\widetilde{x} \cdot j_*\widetilde{x'} = j_*(c_1(T) \cdot \widetilde{x} \cdot \widetilde{x'})$;

(iii) $f^*y \cdot j_*\widetilde{x} = j_*((g^*\Delta^*y) \cdot \widetilde{x})$.

Taking the involution into account, we have split exact sequences

$$0 \to A_k(S) \xrightarrow{\alpha} A_k(\widetilde{S}) \bigoplus A_k(S \times S)^{\mathbb{Z}_2} \xrightarrow{\beta} A_k(\widetilde{S \times S})^{\mathbb{Z}_2} \to 0,$$

thus deciding the homology groups of the Hilbert scheme $S^{[2]}$. Obviously, the homology group $A_k(S^{[2]})$ consists of two parts, one from $A_k(\widetilde{S})$, one from $A_k(S \times S)^{\mathbb{Z}_2}$, identified under α.

The homology classes in $A_k(\widetilde{S})$ are sent in $A_k(\widetilde{S \times S})^{\mathbb{Z}_2}$ by the embedding j of the exceptional divisor and the groups themselves are presented in terms of those of the base (see e.g. P.606 Griffiths&Harris: Principles of Algebraic Geometry).

Proposition 3.2.2. *Let $\zeta = c_1(T)$. Then*

$$A^*(\widetilde{S}) = A^*(S)[\zeta]/(\zeta^2 - c_1(TS)\zeta + c_2(TS)),$$

as graded rings.

The relation between the Gysin map f^* and the proper transform is described in the following proposition, which is modified from Theorem 6.7 in [8].

Proposition 3.2.3. *Let V be a k-dimensional subvariety in $S \times S$, and let $\widetilde{V}$ be the proper transform of V in $\widetilde{S \times S}$. Then*

$$f^*[V] = [\widetilde{V}] + j_*\{c(E) \cap g^*s(V \bigcap \Delta, V)\}_k$$

in $A_k(\widetilde{S \times S})$, where $\{\cdot\}_k$ means taking the degree k part of the class.

Corollary 3.2.4. *(Corollary 6.7.2[8]) With the assumption of the above proposition, when $dim V \bigcap \Delta \leq k-2$,*

$$f^*[V] = [\widetilde{V}].$$

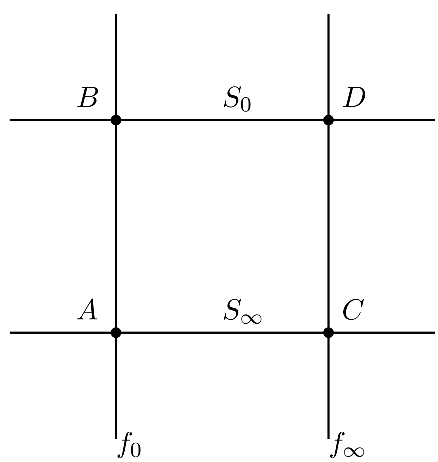
In order to apply the results above to our case of Hirzebruch surface F_a, we need to review the construction of its homology groups or Chow groups[11].

Let

$$\pi : F_a = Proj(S^\bullet \mathcal{F}) \to \mathbb{P}^1$$

be the projection, where $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$. The surjective projection $\mathcal{F} \to \mathcal{O}_{\mathbb{P}^1}(-a)$ determines a section of π. We call the image of this section the ∞-section, denoted as S_∞. It is isomorphic to $\mathbb{P}^1$ and forms a divisor in F_a. Note that over F_a, the twisting sheaf $\mathcal{O}_{F_a}(1) \cong \mathcal{L}(S_\infty)$. We call $(0,1)$ the 0-point and $(1,0)$ the ∞-point in $\mathbb{P}^1$, and call the image points of the 0-point and the ∞-point under the section A and C respectively. The projection $\mathcal{F} \to \mathcal{O}_{\mathbb{P}^1} \to 0$ also determines a section of π, which is named as the 0-section. We refer to its image as S_0. We call the image points of the 0-point and the ∞-point in $\mathbb{P}^1$ under this section B and D respectively. We denote the fibre over the 0-point by f_0 and the fibre over the ∞-point by f_∞. These special divisors and their geometric configuration can be illustrated in the following picture:

Figure 3.1: Special Divisors in F_a



It is a fact that any two fibres of π are numerically equivalent. We also have the numerical equivalence relation

$$S_0 \equiv S_\infty + af.$$

Then the homology groups of F_a are given as follows:

$$A_0(F_a) = \mathbb{Z}pt;$$

$$A_1(F_a) = \mathbb{Z}S \oplus \mathbb{Z}f;$$

$$A_2(F_a) = \mathbb{Z}F_a,$$

where pt, here and in the sequel when not designated, is any one of the fixed or arbitrary points on the relevant spaces, which assumes transversality as much as possible when we do intersection theory, S means either one of S_0 and S_∞, and f means either one of f_0 and f_∞. We keep the freedom as to choose which one of them when needed. The various intersection products are described as follows:

$$S_0^2 = a, \;\; S_\infty^2 = -a, \;\; S_0 \cdot S_\infty = 0, \;\; f^2 = 0, \;\; S \cdot f = 1.$$

Also, the canonical divisor $K \equiv -2S_\infty - (2+a)f$, with $K^2 = 8$. Notice that $c_1(TF_a) = -K = 2S_\infty + (2+a)f = 2S_0 + (2-a)f$. Note that these cycles are smooth submanifolds. This nice property gives rise to smoothness of representative cycles for the Hilbert schemes, which offers us much convenience later in our computations.

When we think about applying Proposition 3.2.1 to F_a, there are two different types of homology classes. The classes in the first type are just the pullbacks under g of the homology classes of F_a, from which we obtain $P(TF_a|_{pt}), P(TF_a|_S), P(TF_a|_f)$, and $\widetilde{F_a}$ itself, where here and in the following when we don't designate subscript for S or f, we mean either one of the choices. We need to take the cap products of ζ with the classes in the first type to get the classes in the second type.

Lemma 3.2.5. *There are relations among the various cycles:*

(1)$P(TF_a|_{S_0}) = P(TF_a|_{S_\infty}) + aP(TF_a|_f)$;

(2)$\zeta \cap P(TF_a|_{S_\infty}) = (2-a)P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_{S_\infty})$;

(3)$\zeta \cap P(TF_a|_{S_0}) = (2+a)P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_{S_0})$;

(4)$\zeta \cap P(TF_a|_f) = 2P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_f)$;

(5)$\zeta \cap P(TF_a) = 2P(TF_a|_{S_\infty}) + (2+a)P(TF_a|_f) - c_1(E) \cap P(TF_a)$.

Proof. (1) is easy because $S_0 \equiv S_\infty + af$. We get the relation by pulling this identity to $\widetilde{S}$ via g. Since

$$c_1(g^*(TF_a)) = c_1(T) + c_1(E) = \zeta + c_1(E),$$

we have

$$\zeta \cap P(TF_a|_{S_\infty}) = c_1(g^*(TF_a)) \cap P(TF_a|_{S_\infty}) - c_1(E) \cap P(TF_a|_{S_\infty}).$$

On the right-hand side, $c_1(E) \cap P(TF_a|_{S_\infty})$ is identified with $\Delta_* S_\infty$ in $A_1(F_a \times F_a)$, so we just need to figure out the first term. Now since

$$c_1(g^*(TF_a)) = g^* c_1(TF_a),$$

and

$$c_1(TF_a) = -K \equiv 2S_\infty + (2+a)f \equiv 2S_0 + (2-a)f,$$

we have

$$\begin{aligned} c_1(g^*(TF_a)) \cap P(TF_a|_{S_\infty}) &= g^*(2S_0 + (2-a)f) \cap P(TF_a|_{S_\infty}) \\ &= (2-a)P(TF_a|_f) \cap P(TF_a|_{S_\infty}) \\ &= (2-a)P(TF_a|_{pt}), \end{aligned}$$

because the intersection $P(TF_a|_f) \cap P(TF_a|_{S_\infty})$ is proper. So

$$\zeta \cap P(TF_a|_{S_\infty}) = (2-a)P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_{S_\infty}).$$

This is (2). For (4), in the same way,

$$\begin{aligned}
\zeta \cap P(TF_a|_f) &= c_1(g^*(TF_a)) \cap P(TF_a|_f) - c_1(E) \cap P(TF_a|_f) \\
&= g^*(2S_0 + (2+a)f_0) \cap P(TF_a|_{f_\infty}) - c_1(E) \cap P(TF_a|_f) \\
&= 2P(TF_a|_{S_\infty}) \cap P(TF_a|_{f_\infty}) - c_1(E) \cap P(TF_a|_f) \\
&= 2P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_f).
\end{aligned}$$

If we plug (1) in (2) and make use of (4), we get (3). Finally,

$$\begin{aligned}
\zeta \cap P(TF_a) &= c_1(g^*(TF_a)) \cap P(TF_a) - c_1(E) \cap P(TF_a) \\
&= g^*(2S_\infty + (2+a)f_0) \cap P(TF_a) - c_1(E) \cap P(TF_a) \\
&= (2P(TF_a|_{S_\infty}) + (2+a)P(TF_a|_f)) \cap P(TF_a) - c_1(E) \cap P(TF_a) \\
&= 2P(TF_a|_{S_\infty}) + (2+a)P(TF_a|_f) - c_1(E) \cap P(TF_a).
\end{aligned}$$

□

From this lemma, we can find a new set of generators for the homology groups as follows:

$$\begin{aligned}
&A_0(\widetilde{F_a}) = \mathbb{Z}pt; \\
&A_1(\widetilde{F_a}) = \mathbb{Z}P(TF_a|_{pt}) \oplus \mathbb{Z}c_1(E) \cap P(TF_a|_{S_\infty}) \oplus \mathbb{Z}c_1(E) \cap P(TF_a|_f); \\
&A_2(\widetilde{F_a}) = \mathbb{Z}P(TF_a|_S) \oplus \mathbb{Z}P(TF_a|_f) \oplus \mathbb{Z}c_1(E) \cap P(TF_a); \\
&A_3(\widetilde{F_a}) = \mathbb{Z}\widetilde{F_a}.
\end{aligned}$$

The relation $\zeta^2 = c_1(TF_a)\zeta - c_2(TF_a)$ can be modified to produce a relation for the new generators. First, $c_2(TF_a) = 4$. On the other hand,

$$\begin{aligned}
c_1(TF_a)\zeta &= (2S_\infty + (2+a)f)\zeta \\
&= 2\zeta \cap P(TF_a|_{S_\infty}) + (2+a)\zeta \cap P(TF_a|_f) \\
&= 8P(TF_a|_{pt}) - 2c_1(E) \cap P(TF_a|_{S_\infty}) - (2+a)c_1(E) \cap P(TF_a|_f),
\end{aligned}$$

so,

$$\zeta^2 = 4P(TF_a|_{pt}) - 2c_1(E) \cap P(TF_a|_{S_\infty}) - (2+a)c_1(E) \cap P(TF_a|_f).$$

Applying $\zeta = c_1(g^*(TF_a)) - c_1(E)$, we get

$$\begin{aligned}
\zeta^2 &= (c_1(g^*(TF_a))^2 - 2c_1(E)c_1(g^*(TF_a)) + (c_1(E))^2 \\
&= K^2 P(TF_a|_{pt}) - 2c_1(E) \cap g^*(2S_\infty + (2+a)f) + (c_1(E))^2 \\
&= 8P(TF_a|_{pt}) - 4c_1(E) \cap P(TF_a|_{S_\infty}) - (4+2a)c_1(E) \cap P(TF_a|_f) + (c_1(E))^2,
\end{aligned}$$

where $c_1(E)$ can be treated as a homological cycle by duality, or in other words, $c_1(E)$ is identified with $c_1(E) \cap P(TF_a)$. So comparing the above two identities for ζ^2, we obtain relation

$$\begin{aligned}
(c_1(E) \cap P(TF_a))^2 = -4P(TF_a|_{pt}) &+ 2c_1(E) \cap P(TF_a|_{S_\infty}) \\
&+ (2+a)c_1(E) \cap P(TF_a|_f).
\end{aligned}$$

Once the generators of homology groups of F_a have been chosen, we can take a set of generators for $A_k(F_a \times F_a)^{\mathbb{Z}_2}$ as follows:

$$\begin{aligned}
A_0(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}pt; \\
A_1(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(S_\infty \times pt + pt \times S_\infty) \oplus \mathbb{Z}(f \times pt + pt \times f); \\
A_2(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(S_0 \times S_\infty + S_\infty \times S_0) \oplus \mathbb{Z}(f_0 \times f_\infty + f_\infty \times f_0) \\
&\quad \oplus \mathbb{Z}(S_\infty \times f + f \times S_\infty) \oplus \mathbb{Z}(F_a \times pt + pt \times F_a); \\
A_3(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(F_a \times S_\infty + S_\infty \times F_a) \oplus \mathbb{Z}(F_a \times f + f \times F_a); \\
A_4(F_a \times F_a)^{\mathbb{Z}_2} &= \mathbb{Z}(F_a \times F_a).
\end{aligned}$$

Combining the analysis for the homology groups of two part $A_*(\widetilde{F_a})$ and $A_*(F_a \times F_a)^{\mathbb{Z}_2}$, we can find a set of generators for $A_*(\widetilde{F_a \times F_a})^{\mathbb{Z}_2}$. Under the identification of α in Proposition 3.2.1, $c_1(E) \cap P(TF_a|_{S_\infty})$ is identified with $\Delta_*(S_\infty)$, $c_1(E) \cap P(TF_a|_f)$ is identified with $\Delta_*(f)$, and

$c_1(E) \cap P(TF_a)$ is identified with $\Delta_*(F_a)$. But in $A_k(F_a \times F_a)^{\mathbb{Z}_2}$,

$$\Delta_*(S_\infty) = S_\infty \times pt + pt \times S_\infty,$$
$$\Delta_*(f) = f \times pt + pt \times f,$$
$$\Delta_*(F_a) = F_a \times pt + pt \times F_a + S_\infty \times f + f \times S_\infty + \frac{a}{2}(f_0 \times f_\infty + f_\infty \times f_0),$$

so we can take the generators in $A_*(\widetilde{F_a \times F_a})^{\mathbb{Z}_2}$ either from $A_*(\widetilde{F_a})$ via j_* or from $A_*(F_a \times F_a)^{\mathbb{Z}_2}$ via f^*:

$$\alpha_0 = 2pt, \quad \forall pt \in \widetilde{F_a};$$
$$\alpha_1 = 2P(TF_a|pt), \quad \alpha_2 = S_\infty \times pt + pt \times S_\infty, \quad \alpha_3 = f \times pt + pt \times f;$$
$$\alpha_4 = 2P(TF_a|_{S_\infty}), \quad \alpha_5 = 2P(TF_a|_f), \quad \alpha_6 = S_0 \times S_\infty + S_\infty \times S_0,$$
$$\alpha_7 = f_0 \times f_\infty + f_\infty \times f_0, \alpha_8 = S_\infty \times f + f \times S_\infty, \alpha_9 = F_a \times pt + pt \times F_a;$$
$$\alpha_{10} = 2P(TF_a), \quad \alpha_{11} = F_a \times S_\infty + S_\infty \times F_a, \quad \alpha_{12} = F_a \times f + f \times F_a;$$
$$\alpha_{13} = F_a \times F_a,$$

where we omit the symbols j_* and f^*. Define $\beta_i = (\phi^*)^{-1}\alpha_i = \frac{1}{2}\phi_*\alpha_i$ for each i. Then the homology groups of the Hilbert scheme $F_a^{[2]}$ can be expressed in terms of these generators:

$A_0(F_a^{[2]}) = \mathbb{Z}\beta_0;$

$A_1(F_a^{[2]}) = \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2 \oplus \mathbb{Z}\beta_3;$

$A_2(F_a^{[2]}) = \mathbb{Z}\beta_4 \oplus \mathbb{Z}\beta_5 \oplus \mathbb{Z}\beta_6 \oplus \mathbb{Z}\beta_7 \oplus \mathbb{Z}\beta_8 \oplus \mathbb{Z}\beta_9;$

$A_3(F_a^{[2]}) = \mathbb{Z}\beta_{10} \oplus \mathbb{Z}\beta_{11} \oplus \mathbb{Z}\beta_{12};$

$A_4(F_a^{[2]}) = \mathbb{Z}\beta_{13}.$

Notice that under the Gysin map f^*, $\alpha_2, \alpha_3, \alpha_6, \alpha_7, \alpha_8$ and α_9 are mapped to their proper transformations by Corollary 3.2.4, so when followed by the inverse of ϕ^* are sent to smooth submanifolds in $F_a^{[2]}$. This means $\beta_2, \beta_3, \beta_6, \beta_7, \beta_8$ and β_9 are smooth cycles in $F_a^{[2]}$. Of course, the cycles β_1, β_4 and β_5 are obviously smooth in $F_a^{[2]}$. As we said this is a nice property of these cycles. We'll refer these representatives of the classes as standard ones when we use them in computations in Chapter 4 although we have some ambiguities in that some of them have uncertain f's and pt's, but we will designate them when necessary.

The intersection products of the cycles of complementary dimensions are listed in the following tables:

	β_1	β_2	β_3
β_{10}	-2	0	0
β_{11}	0	$-a$	1
β_{12}	0	1	0

	β_4	β_5	β_6	β_7	β_8	β_9
β_4	$2a$	-2	0	0	0	0
β_5	-2	0	0	0	0	0
β_6	0	0	$-a^2$	2	$-a$	0
β_7	0	0	2	0	0	0
β_8	0	0	$-a$	0	1	0
β_9	0	0	0	0	0	1

Other intersection products are computed as follows:

$$
\begin{aligned}
\beta_4 \cdot \beta_{10} &= \frac{1}{2}\phi_*(\alpha_4 \cdot \alpha_{10}) = \frac{1}{2}\phi_*(\zeta \cap 4P(TF_a|_{S_\infty})) \\
&= 2\phi_*((2-a)P(TF_a|_{pt}) - c_1(E) \cap P(TF_a|_{S_\infty})) \\
&= 2\phi_*((2-a)P(TF_a|_{pt}) - \Delta_* S_\infty) \\
&= 2(2-a)\beta_1 - 4\beta_2, \\
\beta_4 \cdot \beta_{11} &= \frac{1}{2}\phi_*(\alpha_4 \cdot \alpha_{11}) \\
&= \phi_*(P(TF_a|_{S_\infty}) \cdot F_a \times S_\infty + P(TF_a|_{S_\infty}) \cdot S_\infty \times F_a) \\
&= \phi_*(-2aP(TF_a|_{pt})) = -2a\beta_1, \\
\beta_4 \cdot \beta_{12} &= \frac{1}{2}\phi_*(\alpha_4 \cdot \alpha_{12}) \\
&= \phi_*(P(TF_a|_{S_\infty}) \cdot F_a \times f + P(TF_a|_{S_\infty}) \cdot f \times F_a) \\
&= \phi_*(2P(TF_a|_{pt})) = 2\beta_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\beta_5 \cdot \beta_{10} = 4\beta_1 - 4\beta_3, && \beta_5 \cdot \beta_{11} = 2\beta_1, \\
&\beta_5 \cdot \beta_{12} = 0, && \beta_6 \cdot \beta_{10} = 0, \\
&\beta_6 \cdot \beta_{11} = -a\beta_2 - a^2\beta_3, && \beta_6 \cdot \beta_{12} = 2\beta_2 + a\beta_3, \\
&\beta_7 \cdot \beta_{10} = 0, && \beta_7 \cdot \beta_{11} = 2\beta_3, \\
&\beta_7 \cdot \beta_{12} = 0, && \beta_8 \cdot \beta_{10} = 2\beta_1, \\
&\beta_8 \cdot \beta_{11} = \beta_2 - a\beta_3, && \beta_8 \cdot \beta_{12} = \beta_3, \\
&\beta_9 \cdot \beta_{10} = 2\beta_1, && \beta_9 \cdot \beta_{11} = \beta_2, \\
&\beta_9 \cdot \beta_{12} = \beta_3.
\end{aligned}
$$

Also,

$$\begin{aligned}
&\beta_{10}^2 = 4\beta_4 + 2(2+a)\beta_5 - 2a\beta_7 - 4\beta_8 - 4\beta_9,\\
&\beta_{10}\cdot\beta_{11} = 2\beta_4, \qquad\qquad \beta_{10}\cdot\beta_{12} = 2\beta_5,\\
&\beta_{11}^2 = \beta_6 - a\beta_8 - a\beta_9, \qquad \beta_{11}\cdot\beta_{12} = \beta_8 + \beta_9,\\
&\beta_{12}^2 = \beta_7.
\end{aligned}$$

From the last six lines of identities, we find that the Chow ring of $F^{[2]}$ is generated by $\beta_{10}, \beta_{11}, \beta_{12}, \beta_9$ and the other basis elements are expressed as

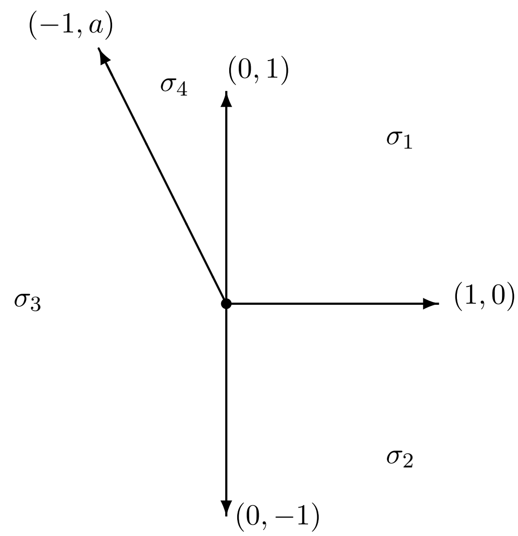
$$\begin{aligned}
&\beta_1 = \frac{1}{2}\beta_9\beta_{10}, \qquad \beta_2 = \beta_9\beta_{11}, \qquad \beta_3 = \beta_9\beta_{12},\\
&\beta_4 = \frac{1}{2}\beta_{10}\beta_{11}, \quad \beta_5 = \frac{1}{2}\beta_{10}\beta_{12}, \quad \beta_6 = \beta_{11}^2 + \beta_{11}\beta_{12},\\
&\beta_7 = \beta_{12}^2, \qquad\quad \beta_8 = \beta_{11}\beta_{12} - \beta_9.
\end{aligned}$$

Plugging these expressions in the remaining identities including the above two product tables, we get various equations on the generators. By ruling out the redundant ones, we conclude the Chow ring is presented by the following relations:

$$\begin{aligned}
&\beta_9\beta_{11}\beta_{12} = \beta_9^2,\\
&\beta_{10}\beta_{12}^2 = 0,\\
&\beta_{12}^3 = 0,\\
&\beta_{11}\beta_{12}^2 - 2\beta_9\beta_{12} = 0,\\
&\beta_{10}\beta_{11}^2 + 2a\beta_9\beta_{10} = 0,\\
&\beta_{10}\beta_{11}\beta_{12} - 2\beta_9\beta_{10} = 0,\\
&\beta_{11}^2\beta_{12} - 2\beta_9\beta_{11} + a\beta_9\beta_{12} = 0,\\
&\beta_{11}^3 + 3a\beta_9\beta_{11} = 0,\\
&\beta_{10}^2 - 2\beta_{10}\beta_{11} - (2+a)\beta_{10}\beta_{12} + 2a\beta_{12}^2 + 4\beta_{11}\beta_{12} = 0.
\end{aligned}$$

3.3 Torus Action on Hilbert Schemes

As a toric variety, F_a can be constructed from the fan in the following picture(See Fulton, Introduction to toric varieties):



The four affine varieties are

$$U_{\sigma_1} = Spec\mathbb{C}[x, y], \quad U_{\sigma_2} = Spec\mathbb{C}[x, y^{-1}],$$
$$U_{\sigma_3} = Spec\mathbb{C}[x^{-1}, x^{-a}y^{-1}], \quad U_{\sigma_4} = Spec\mathbb{C}[x^{-1}, x^a y].$$

Note that the origins of the four affine planes correspond to the points we named A, B, D and C respectively, in Figure 3.1. Now $\mathbb{C}^{*2}$ acts on F_a by acting on the variables in the following way:

$$(\lambda, \mu)(x, y) = (\lambda^{-1}x, \mu^{-1}y),$$
$$(\lambda, \mu)(x, y^{-1}) = (\lambda^{-1}x, \mu y^{-1}),$$
$$(\lambda, \mu)(x^{-1}, x^{-a}y^{-1}) = (\lambda x^{-1}, \lambda^a \mu x^{-a} y^{-1}),$$
$$(\lambda, \mu)(x^{-1}, x^a y) = (\lambda x^{-1}, \lambda^{-a}\mu^{-1}x^a y).$$

Dually, $\mathbb{C}^{*2}$ acts on each coordinate piece with opposite weights. This action in turn induces an action on the Hilbert scheme $F_a^{[2]}$. A moment of thinking shows that the representative cycles $\beta_1, \cdots, \beta_{13}$ are invariant under the torus action.

It is easy to see that around each fixed point in F_a, there are two fixed points, one corresponding to the direction of the ∞-section or the 0-section, one corresponding to the direction of the fibre through it. We denote them using subscripts "1" or "2", respectively. So we have eight of them, $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$. For the computational purpose, we need to find the weights of the torus action at these fixed points.

We start with A_1. In the first affine plane, A_1 is represented by the ideal (x^2, y). The full deformation of (x^2, y) in $F_a^{[2]}$ is described by $(x^2 + \varepsilon_1 x + \varepsilon_2, y + \varepsilon_3 x + \varepsilon_4)$. So there are four curves passing through A_1 given by families of ideals in $\mathbb{C}[x, y]$: $I_1(\varepsilon) = (x^2 + \varepsilon x, y), I_2(\varepsilon) = (x^2 + \varepsilon, y), I_3(\varepsilon) = (x^2, y + \varepsilon x)$, and $I_4(\varepsilon) = (x^2, y + \varepsilon)$.

Lemma 3.3.1. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at A_1 are $\lambda, 2\lambda, \mu - \lambda$, and μ.*

Proof. We have

$$
\begin{aligned}
(\lambda, \mu) I_1(\varepsilon) &= (\lambda, \mu)(x^2 + \varepsilon x, y) = (\lambda^{-2} x^2 + \varepsilon \lambda^{-1} x, \mu^{-1} y) \\
&= (x^2 + \varepsilon \lambda x, y) = I_1(\varepsilon \lambda).
\end{aligned}
$$

So the weight on the tangent direction of this curve is λ.

Similarly,

$$
\begin{aligned}
(\lambda, \mu) I_2(\varepsilon) &= (\lambda^{-2} x^2 + \varepsilon, \mu^{-1} y) = I_2(\varepsilon \lambda^2), \\
(\lambda, \mu) I_3(\varepsilon) &= (\lambda^{-2} x^2, \mu^{-1} y + \varepsilon \lambda^{-1} x) = I_3(\varepsilon \lambda^{-1} \mu), \\
(\lambda, \mu) I_4(\varepsilon) &= (\lambda^{-2} x^2, \mu^{-1} y + \varepsilon) = I_4(\varepsilon \mu).
\end{aligned}
$$

So the weights on the tangent directions of these curves are $2\lambda, \mu - \lambda, \mu$. $\square$

In the same vein, there are four curves passing through A_2, which is represented by the ideal (x, y^2). They are families of ideals $I_1(\varepsilon) = (x + \varepsilon, y^2), I_2(\varepsilon) = (x, y^2 + \varepsilon y), I_3(\varepsilon) = (x + \varepsilon y, y^2)$, and

$I_4(\varepsilon) = (x, y^2 + \varepsilon)$.

Lemma 3.3.2. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at A_2 are $\lambda, \mu, \lambda - \mu$, and 2μ.*

Proof. Similar to that of Lemma 1. $\square$

Four curves traverse $B_1 : I_1(\varepsilon) = (x^2 + \varepsilon, y^{-1}), I_2(\varepsilon) = (x^2 + \varepsilon x, y^{-1}), I_3(\varepsilon) = (x^2, y^{-1} + \varepsilon)$, and $I_4(\varepsilon) = (x^2, y^{-1} + \varepsilon x)$.

Lemma 3.3.3. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at B_1 are $2\lambda, \lambda, -\mu$ and $-\lambda - \mu$.*

For B_2, four curves go through it: $I_1(\varepsilon) = (x + \varepsilon, y^{-2}), I_2(\varepsilon) = (x, y^{-2} + \varepsilon), I_3(\varepsilon) = (x, y^{-2} + \varepsilon y^{-1})$, and $I_4(\varepsilon) = (x + \varepsilon y^{-1}, y^{-2})$.

Lemma 3.3.4. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at B_2 are $\lambda, -2\mu, -\mu$ and $\lambda + \mu$.*

There are four curves traversing $C_1 : I_1(\varepsilon) = (x^{-2} + \varepsilon, x^a y), I_2(\varepsilon) = (x^{-2} + \varepsilon x^{-1}, x^a y), I_3(\varepsilon) = (x^{-2}, x^a y + \varepsilon)$, and $I_4(\varepsilon) = (x^{-2}, x^a y + \varepsilon x^{-1})$.

Lemma 3.3.5. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at C_1 are $-2\lambda, -\lambda, a\lambda + \mu$, and $(a+1)\lambda + \mu$.*

At last for C_2, four curves go through it: $I_1(\varepsilon) = (x^{-1} + \varepsilon, x^{2a} y^2), I_2(\varepsilon) = (x^{-1}, x^{2a} y^2 + \varepsilon), I_3(\varepsilon) = (x^{-1}, x^{2a} y^2 + \varepsilon x^a y)$, and $I_4(\varepsilon) = (x^{-1} + \varepsilon x^a y, x^{2a} y^2)$.

Lemma 3.3.6. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at C_2 are $-\lambda, 2a\lambda + 2\mu, a\lambda + \mu$, and $-(a+1)\lambda - \mu$.*

Four curves pass through $D_1 : I_1(\varepsilon) = (x^{-2} + \varepsilon x^{-1}, x^{-a} y^{-1}), I_2(\varepsilon) = (x^{-2} + \varepsilon, x^{-a} y^{-1}), I_3(\varepsilon) = (x^{-2}, x^{-a} y^{-1} + \varepsilon x^{-1})$, and $I_4(\varepsilon) = (x^{-2}, x^{-a} y^{-1} + \varepsilon)$.

Lemma 3.3.7. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at D_1 are $-\lambda, -2\lambda, (1-a)\lambda - \mu$, and $-a\lambda - \mu$.*

Four curves pass through $D_2 : I_1(\varepsilon) = (x^{-1}+\varepsilon, x^{-2a}y^{-2}), I_2(\varepsilon) = (x^{-1}, x^{-2a}y^{-2}+\varepsilon x^{-a}y^{-1}), I_3(\varepsilon) = (x^{-1} + \varepsilon x^{-a}y^{-1}, x^{-2a}y^{-2})$, and $I_4(\varepsilon) = (x^{-1}, x^{-2a}y^{-2} + \varepsilon)$.

Lemma 3.3.8. *The weights of the $\mathbb{C}^{*2}$-action on the tangent space of $F_a^{[2]}$ at D_2 are $-\lambda, -a\lambda - \mu, (a-1)\lambda + \mu$, and $-2a\lambda - 2\mu$.*

These eight points form the so-called nonreduced fixed points under the torus action on the Hilbert scheme $F_a^{[2]}$. The other fixed points are composed of pairs of distinct fixed points under the torus action on F_a itself, which we denote by juxtaposing points. As we know, these fixed points on F_a are A, B, C, and D, so we have six of this type of fixed points on $F_a^{[2]}$: $(AB), (AC), (AD), (BC), (BD)$, and (CD). The weights of the $\mathbb{C}^{*2}$-action on the tangent spaces of F_a at fixed points are: λ, μ at A; $\lambda, -\mu$ at B; $-\lambda, a\lambda + \mu$ at C; $-\lambda, -a\lambda - \mu$ at D. The weights of the $\mathbb{C}^*$-action on the tangent spaces of $F_a^{[2]}$ at this type of fixed points are just putting together the weights at the different points. So we have

Lemma 3.3.9. *The weights of the $\mathbb{C}^*$-action on the tangent spaces of $F_a^{[2]}$ at these six fixed points are:*

$(1)(AB) : \lambda, \mu, \lambda, -\mu;$

$(2)(AC) : \lambda, \mu, -\lambda, a\lambda + \mu;$

$(3)(AD) : \lambda, \mu, -\lambda, -a\lambda - \mu;$

$(4)(BC) : \lambda, -\mu, -\lambda, a\lambda + \mu;$

$(5)(BD) : \lambda, -\mu, -\lambda, -a\lambda - \mu;$

$(6)(CD) : -\lambda, a\lambda + \mu, -\lambda, -a\lambda - \mu.$

After we determine all the weights of the torus action on the Hilbert scheme, we can compute the intersection numbers of the curve classes with its anticanonical bundle, which is needed to decide the virtual dimension of the moduli space of stable maps.

Lemma 3.3.10. $c_1(TF_a^{[2]}) \cdot \beta_1 = 0,\ \ c_1(TF_a^{[2]}) \cdot \beta_2 = 2 - a,\ \ c_1(TF_a^{[2]}) \cdot \beta_3 = 2.$

Proof. Take the image of $P(TF_a|_A)$ in the Hilbert scheme under $(\phi^*)^{-1}$ in §3.2 as the submanifold representing β_1. The two fixed points in this cycle are A_1 and A_2, around which $\mathbb{C}^{*2}$ acts as it does on $I_3(\varepsilon)$ in Lemma 3.3.1 and on $I_3(\varepsilon)$ in Lemma 3.3.2, respectively. So the weights on the tangent spaces at the two points are $\lambda-\mu$ and $\mu-\lambda$. By the localization formula,

$$\begin{aligned}
c_1(TF_a^{[2]})\cdot\beta_1 &= \int_{\beta_1} c_1(TF_a^{[2]})\\
&= \frac{-\lambda-2\lambda+\lambda-\mu-\mu}{\lambda-\mu}+\frac{-\lambda-\mu+\mu-\lambda-2\mu}{\mu-\lambda}\\
&= 0.
\end{aligned}$$

We take the image of $S_\infty\times B+B\times S_\infty$ in $F_a^{[2]}$ as the representative of β_2. The two fixed points are AB and CB. From the discussion before Lemma 3.3.9, $\mathbb{C}^{*2}$ acts on the tangent spaces at these points with weights λ and $-\lambda$. So applying localization formula, we get

$$\begin{aligned}
c_1(TF_a^{[2]})\cdot\beta_2 &= \int_{\beta_2} c_1(TF_a^{[2]})\\
&= \frac{\lambda+\mu+\lambda-\mu}{\lambda}+\frac{\lambda-\mu-\lambda+a\lambda+\mu}{-\lambda}\\
&= 2-a.
\end{aligned}$$

We take the image of $f_0\times C+C\times f_0$ as the representative of β_3, which has two fixed points AC and BC. The weights of the torus action on the tangent spaces at these two points are μ and $-\mu$. So by the localization formula again,

$$\begin{aligned}
c_1(TF_a^{[2]})\cdot\beta_3 &= \int_{\beta_3} c_1(TF_a^{[2]})\\
&= \frac{\lambda+\mu-\lambda+a\lambda+\mu}{\mu}+\frac{\lambda-\mu-\lambda+a\lambda+\mu}{-\mu}\\
&= 2.
\end{aligned}$$

□

Notice that when $a=1$, these intersection numbers are nonnegative. This is a precious property for a variety. When $a=2$, two generators of curve classes have a trivial intersection with the anticanonical bundle; when $a\geq 2$, we begin to have negative intersections from one generator. In general, computations of Gromov-witten invariants become more difficult if we have more negative intersections because the formula for the virtual dimension places less restrictions on degrees of cohomological classes with nonzero invariants. This is the reason why we'll concentrate on the case $a = 1$ later.

3.4 Invariant Curves

For purpose of applying virtual localization to calculate Gromov-witten invariants, we have to determine all the invariant curves of the torus action. In this section and in the following, when there is no danger of confusion, we'll use the terms "line" and "$\mathbb{P}^1$" interchangeably when we talk about curves of genus zero. To find them, we make use of the blowup construction of the Hilbert scheme set up in section 3.2.

Let $f : \widetilde{F_a \times F_a} \to F_a \times F_a$ be the blowup of the product $F_a \times F_a$ along its diagonal Δ, with the exceptional divisor $\widetilde{F_a}$, which is the projective bundle $P(TF_a)$. Then $F_a^{[2]}$ is the $\mathbb{Z}_2$-quotient of the blowup. Since this quotient map is equivariant, it suffices for us to find the invariant curves in $\widetilde{F_a \times F_a}$.

First of all, an invariant curve either is completely contained in $\widetilde{F}_a$ or intersects this exceptional divisor in only finitely many points, where the word "finitely many" could mean zero.

We first consider the case when the invariant curve is contained in $\widetilde{F}_a$. Since the projection $P(TF_a) \to F_a$ is equivariant, this invariant curve is mapped to either a fixed point or an invariant curve in F_a. When it is mapped to a fixed point, it must be the fibre curve of the projective bundle over this fixed point. So in $F_a^{[2]}$, we get four invariant $\mathbb{P}^1$s, corresponding to four fixed points A, B, C and D. We assign names to these curves by listing their end points. For example, the curve over A as $[A_1, A_2]$, connecting the fixed points A_1 and A_2 in $F_a^{[2]}$. Here we adopt the convention that $[P, Q]$ means either the invariant curve connecting two fixed points P and Q, or the degree class of this curve, depending on the context. Three other such curves are $[B_1, B_2], [C_1, C_2]$ and $[D_1, D_2]$. They are isolated invariant lines.

Now suppose this invariant curve is mapped to an invariant curve in F_a. We only have four invariant curves in F_a, which are S_∞, S_0, f_0 and f_∞. Take the section S_∞ through A and C in F_a as an example. Then this invariant curve must be contained in $P(TF_a|_{S_\infty})$, but $TF_a|_{S_\infty} = TS_\infty \oplus N_{S_\infty|F_a}$, where $N_{S_\infty|F_a}$ represents fibre directions of S_∞ in F_a, so $P(TF_a|_{S_\infty})$ is also a rational ruled surface. With induced action on this surface, the two sections corresponding to the tangent directions and fibre directions of S_∞ in F_a respectively, are invariant lines. One is $[A_1, C_1]$, representing the tangent directions; the other is $[A_2, C_2]$, representing the fibre directions. Similarly, we have the invariant lines $[A_1, B_1], [A_2, B_2], [B_1, D_1], [B_2, D_2], [C_1, D_1]$ and $[C_2, D_2]$, corresponding to either tangent directions or fibre or normal directions of three other invariant lines in F_a.

Assume an invariant curve only intersects the exceptional divisor in finitely many points. Then the blow-down map f composed with the two projections from $F_a \times F_a$ to F_a gives rise to two maps to F_a, which are also equivariant. So as the images of the invariant curve under these two maps, we get either fixed points or invariant lines in F_a. The two images cannot both be points since the curve is not contained in the exceptional divisor. If we have a point and a line which does not pass through the point, we get an isolated invariant line in $F_a^{[2]}$. Considering the $\mathbb{Z}_2$- symmetry, we have eight of them $[AB, AD], [AC, AD], [AB, BC], [BC, BD], [AC, BC], [BC, CD], [AD, BD]$ and $[AD, CD]$. If the point is contained in the line, we get an invariant line with one non-reduced point on it, which is also isolated. They are $[A_1, AC], [A_2, AB], [B_2, AB], [B_1, BD], [C_1, AC], [C_2, CD], [D_1, BD]$ and $[D_2, CD]$.

Suppose both images are lines. First consider the case when they are disjoint. Then they are either the pair S_∞ and S_0 or the pair f_0 and f_∞. In the first case, the $\mathbb{C}^{*2}$-action near $(A, B) \in S_\infty \times S_0$ is described by $(\lambda, \mu)(x, y) = (\lambda x, \lambda y)$, so we have a one-dimensional family of invariant $\mathbb{P}^1$'s connecting AB to CD. Near $(A, C) \in f_0 \times f_\infty$, the action is expressed as $(\lambda, \mu)(x, y) = (\mu x, \lambda^a \mu y)$. Since the two weights are independent, no invariant curve is brought up from this action. So in total, this case makes a single one-dimensional family of invariant lines connecting AB to CD.

Now take the case when the two image lines are distinct but intersect at one fixed point, e.g. S_∞ and f_0. Then $\mathbb{C}^{*2}$-action near $(A, A) \in S_\infty \times f_0$ is expressed as $(\lambda, \mu)(x, y) = (\lambda x, \mu y)$. This action does not produce any invariant curve except the two coordinate lines, whose induced curves in $F_a^{[2]}$ have been discussed above. Other combinations of invariant lines in this case neither produce anything new.

Finally, we are left with the case where the two lines are the same. To understand the situation, we study a concrete example. Let $\mathbb{C}^*$ act on $\mathbb{C}$ with the standard weight, i.e. $\lambda \cdot x = \lambda x$. Then it induces an action on $\mathbb{C}^{[2]} = \mathbb{C}^{(2)}$, which is the symmetric product of $\mathbb{C}$. We define a map $\tau : \mathbb{C}^{(2)} \to \mathbb{C}^2$ by $\tau(x, y) = (x + y, xy)$. It is easy to see that this map is an isomorphism. With this map, $\mathbb{C}^2$ inherits a $\mathbb{C}^*$-action by $\lambda(x,z) = (\lambda x, \lambda^2 z)$ for $(x,z) \in \mathbb{C}^2$. This means the weights of the torus action at the origin is $\lambda, 2\lambda$.

In fact, the map τ extends to an isomorphism from the symmetric product of $\mathbb{P}^1$ to $\mathbb{P}^2$, still denoted as $\tau : (\mathbb{P}^1)^{(2)} \to \mathbb{P}^2$ by $\tau((a,b),(x,y)) = (ay+bx, ax, by)$. It is not hard to see that the image of the diagonal of $(\mathbb{P}^1)^{(2)}$ in $\mathbb{P}^2$ is a conic line but this conic is not isolated as an invariant line. In fact, there is a one-dimensional family of invariant conic lines in $\mathbb{P}^2$, which breaks up to two coordinate lines[16]. We summarize the conclusions in the following

Lemma 3.4.1. *Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ as $\lambda \cdot (x,y) = (\lambda x, y)$. Then it induces an action on $\mathbb{P}^2$ as $\lambda \cdot (x,y,z) = (\lambda x, \lambda^2 y, z)$ via the isomorphism from the symmetric product $(\mathbb{P}^1)^{(2)}$ to $\mathbb{P}^2$. Around the fixed point $(0,0,1) \in \mathbb{P}^2$, this action has weights $\lambda, 2\lambda$; the three coordinate lines are isolated invariant lines and there is a one-dimensional family of invariant lines defined by $x^2 = \mu yz$ with $\mu \in \mathbb{C}^* - 0$, the generic curve of which has class twice the line class in $\mathbb{P}^2$. When $\mu \to 0$, this family approaches the double cover of the coordinate line $\{(0,y,z) : (y,z) \in \mathbb{P}^1\}$ with weight 2λ at $(0,0,1)$; when $\mu \to \infty$, it degenerates to the nodal curve the union of two coordinate lines $\{(x,0,z) : (x,z) \in \mathbb{P}^1\} \cup \{(x,y,0) : (x,y) \in \mathbb{P}^1\}$ with weight λ at $(0,0,1)$ along $\{(x,0,z) : (x,z) \in \mathbb{P}^1\}$.*

Now we apply this picture to the four invariant lines S_∞, S_0, f_0 and f_∞ in F_a. Taking S_∞ as an example, we get an embedding $\mathbb{P}^2$ in $F_a^{[2]}$ and the three coordinate lines in $\mathbb{P}^2$ as the lines $[A_1, AC], [AC, C_1]$ as before and $[A_1, C_1]'$ from A_1 to C_1, which is a different line from $[A_1, C_1]$ discussed before, and a one-dimensional family of invariant lines from A_1 to C_1, the generic curve of which has the double of a line class. Similarly, we have the new isolated invariant lines $[A_2, B_2]'$ from A_2 to B_2, $[B_1, D_1]'$ from B_1 to D_1 and $[C_2, D_2]'$ from C_2 to D_2, and the corresponding one-dimensional families of invariant lines. From the description before this lemma, we realize that the invariant lines $[A_1, C_1], [A_2, B_2], [B_1, D_1]$ and $[C_2, D_2]$ defined before are obtained from diagonals and as such they are not isolated and their classes are double the classes of $[A_1, C_1]', [A_2, B_2]', [B_1, D_1]'$ and $[C_2, D_2]'$ respectively. Also the $\mathbb{C}^*$-action on the ends of $[A_1, C_1]', [A_2, B_2]', [B_1, D_1]'$ and $[C_2, D_2]'$ has twice the weights on the corresponding ends of the non-isolated lines. This is the reason why a pair of a weight and its double always appear at each fixed point exhibited in Lemma 3.3.1 through 3.3.8.

Up till this point, we have found all the isolated invariant curves and one-dimensional families of invariant curves in $F_a^{[2]}$. For the purpose of computations for Gromov-Witten invariants, their classes should be decided.

Lemma 3.4.2. $(i)[A_1,A_2]=[B_1,B_2]=[C_1,C_2]=[D_1,D_2]=\beta_1$;

$(ii)[A_1,AC]=[C_1,AC]=\beta_2-\beta_1$, $[B_1,BD]=[D_1,BD]=\beta_2+a\beta_3-\beta_1$, $[A_2,AB]=[B_2,AB]=[C_2,CD]=[D_2,CD]=\beta_3-\beta_1$;

$(iii)[A_1,C_1]=2\beta_2-2\beta_1$, $[A_2,C_2]=2\beta_2+a\beta_1$, $[A_1,B_1]=[C_1,D_1]=2\beta_3$, $[A_2,B_2]=[C_2,D_2]=2\beta_3-2\beta_1$, $[B_1,D_1]=2\beta_2+2a\beta_3-2\beta_1$, $[B_2,D_2]=2\beta_2+2a\beta_3-a\beta_1$;

$(iv)[A_1,C_1]'=\beta_2-\beta_1$, $[A_2,B_2]'=[C_2,D_2]'=\beta_3-\beta_1$, $[B_1,D_1]'=\beta_2+a\beta_3-\beta_1$.

Proof. (i) Obvious.

(ii) A_1 is connected to AC by the line which is obtained by blowing up the line $S_\infty\times A+A\times S_\infty$ at $A\times A$ in $F_a\times F_a$ and then projecting to $A_*(F_a^{[2]})$ by the inverse of ϕ^*. By Proposition 3.2.3,

$$f^*(S_\infty\times A+A\times S_\infty)=[A_1,AC]+\beta_1,$$

where β_1 comes from the second term in the formula. But $S_\infty\times A+A\times S_\infty$ is rationally equivalent to $S_\infty\times pt+pt\times S_\infty$ in $F_a\times F_a$, where we take pt to be a point off S_∞. So $[A_1,AC]=\beta_2-\beta_1$. Others are similar.

(iii) $[A_1,C_1]$ is the line obtained from the projectivization of the tangent directions of the section S_∞. By Lemma 3.2.5(2),

$$\zeta\cap P(TF_a|_{S_\infty})=(2-a)\beta_1-2\beta_2.$$

Now $TF_a|_{S_\infty}=TS_\infty\oplus N_{S_\infty|F_a}$, in which $TS_\infty=\mathcal{O}(2)$ since $S_\infty=\mathbb{P}^1$ and $N_{S_\infty|F_a}$ is the line bundle of fibre directions of S_∞ in F_a, implying $N_{S_\infty|F_a}=\mathcal{O}(-a)$. This means that

$$\begin{aligned}P(TF_a|_{S_\infty})&=P(\mathcal{O}(2)\oplus\mathcal{O}(-a))=Proj(\mathcal{O}(-2)\oplus\mathcal{O}(a))\\&\cong Proj(\mathcal{O}\oplus\mathcal{O}(-2-a))\end{aligned}$$

is another rational ruled surface, in which $[A_1, C_1]$ is the ∞-section. By Lemma 7.9, Ch.2[11],

$$\zeta \cap P(TF_a|_{S_\infty}) = -[A_1, C_1] - a\beta_1.$$

So, we get $[A_1, C_1] = 2\beta_2 - 2\beta_1$. Clearly, $[A_2, C_2]$ is the 0-section in the ruled surface, which means

$$[A_2, C_2] = [A_1, C_1] + (2 + a)\beta_1 = 2\beta_2 + a\beta_1.$$

Similarly, by Lemma 3.2.5(4), we have

$$\zeta \cap P(TF_a|_f) = 2\beta_1 - 2\beta_3.$$

Now $P(TF_a|_{f_0}) = Tf \oplus N_{f_0|TF_a}$, where again $Tf_0 = T\mathbb{P}^1 = \mathcal{O}(2)$ and $N_{f_0|TF_a}$ is the line bundle of normal directions of f_0 in F_a. This line bundle is trivial since

$$c_1(N_{f_0|TF_a}) = c_1(TF_a|_f) - c_1(Tf_0) = f_0 \cdot (2S_0 + (2 + a)f_0) - 2 = 0.$$

So we have

$$P(TF_a|_{f_0}) = P(\mathcal{O} \oplus \mathcal{O}(2)) = Proj(\mathcal{O} \oplus \mathcal{O}(-2)),$$

which is another ruled surface. Thus we have

$$[A_2, B_2] = -\zeta \cap P(TF_a|_{f_0}) = 2\beta_3 - 2\beta_1,$$

and $[A_1, B_1] = [A_2, B_2] + 2\beta_1 = 2\beta_3$. It is obvious that $[C_1, D_1] = [A_1, B_1]$ and $[C_2, D_2] = [A_2, B_2]$.

Finally, by Lemma 3.2.5(3) and (1), we have

$$\zeta \cap P(TF_a|_{S_0}) = (2 + a)\beta_1 - 2\beta_2 - 2a\beta_3.$$

Because

$$TF_a|_{S_0} = TS_0 \oplus N_{S_0|F_a} = \mathcal{O}(2) \oplus \mathcal{O}(a),$$

we get

$$\begin{aligned} P(TF_a|_{S_0}) &= P(\mathcal{O}(2) \oplus \mathcal{O}(a)) = Proj(\mathcal{O}(-2) \oplus \mathcal{O}(-a)) \\ &\cong Proj(\mathcal{O} \oplus \mathcal{O}(2-a)), \text{if } a \geq 2; Proj(\mathcal{O} \oplus \mathcal{O}(-1)), \text{if } a = 1 \end{aligned}$$

When $a \geq 2$, by Lemma 7.9, Ch.2[11], we have

$$\zeta \cap P(TF_a|_{S_0}) = -[B_2, D_2] + 2\beta_1,$$

so $[B_2, D_2] = 2\beta_2 + 2a\beta_3 - a\beta_1$, and $[B_1, D_1] = [B_2, D_2] + (a-2)\beta_1 = 2\beta_2 + 2a\beta_3 - 2\beta_1$.

When $a = 1$, we get

$$\zeta \cap P(TF_a|_{S_0}) = -[B_1, D_1] + \beta_1.$$

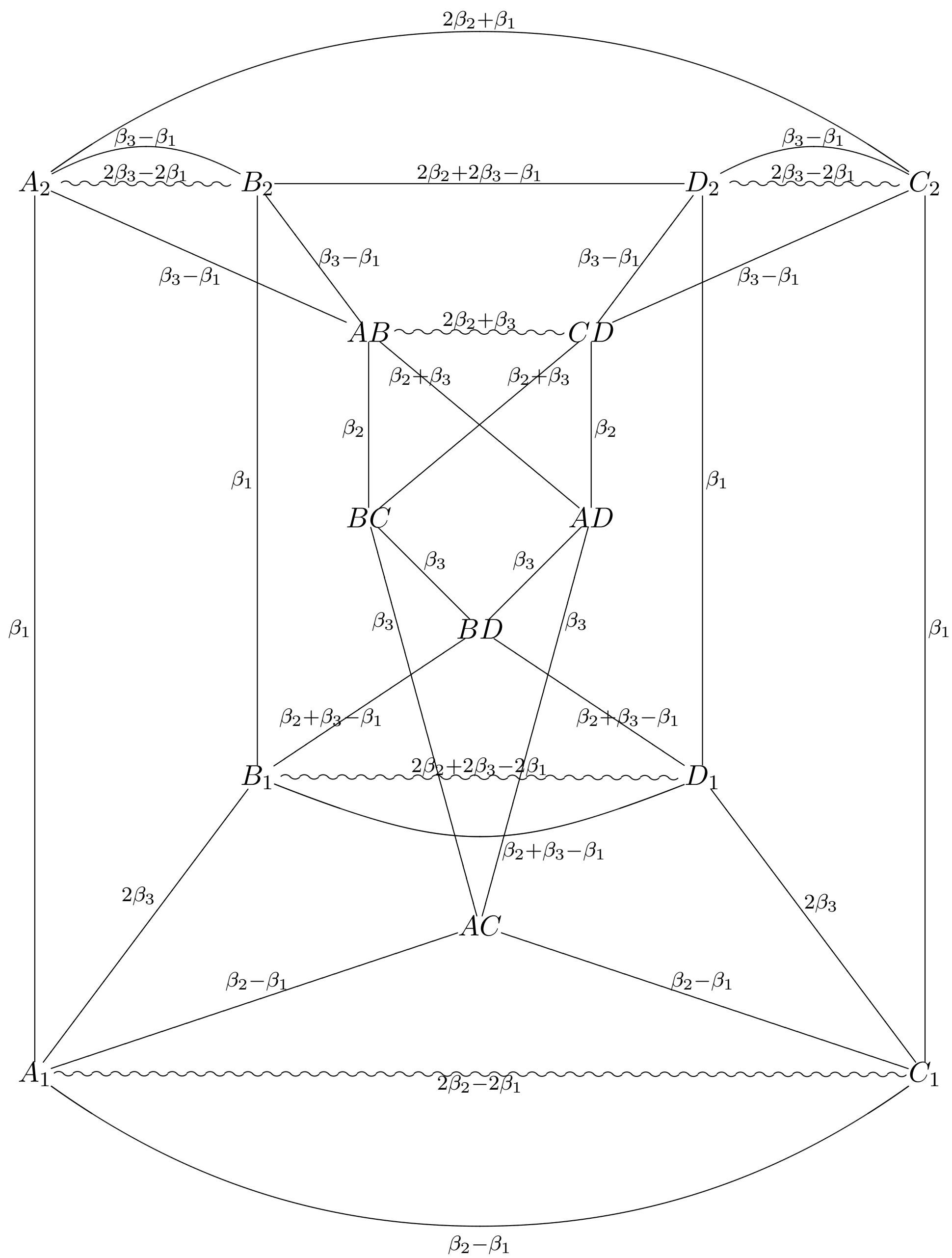
So $[B_1, D_1] = 2\beta_2 + 2\beta_3 - 2\beta_1$ and $[B_2, D_2] = [B_1, D_1] + \beta_1 = 2\beta_2 + 2\beta_3 - \beta_1$.

(iv) We take $[A_1, C_1]'$ as an example. From the discussion after Lemma 3.4.1, the class $[A_1, C_1]'$ is the same as a line class in $\mathbb{P}^2$, one of which is the line $[A_1, AC]$, resulting in the conclusion. □

From this lemma, we know the generic line in the one-dimensional family connecting AB and CD is of class $2\beta_2 + \beta_3$ since as limits it breaks up into the nodal curves the line from AB to AD intersecting the line from AD to CD and the line from AB to BC intersecting the line from BC to CD.

All the isolated invariant lines and one-dimensional families of invariant lines are shown in the following diagram. In this diagram, the isolated invariant lines are depicted by straight or curved lines with their degrees on them, the one-dimensional families of invariant lines are described by wavy lines with the degree of generic lines in the families attached.

Figure 3.2: Invariant Lines



Finally, for the convenience of computational purpose, we make the following diagrams for $F_1^{[2]}$, showing weights at each fixed point connected to four other fixed points along isolated invariant lines drawn from Lemma 3.3.1 through Lemma 3.3.9. In each of the first eight diagrams, we can see a pair of a weight and its double present. These double weights certainly go along the isolated invariant lines connecting two ends.

Figure 3.3: Weights At Fixed Points

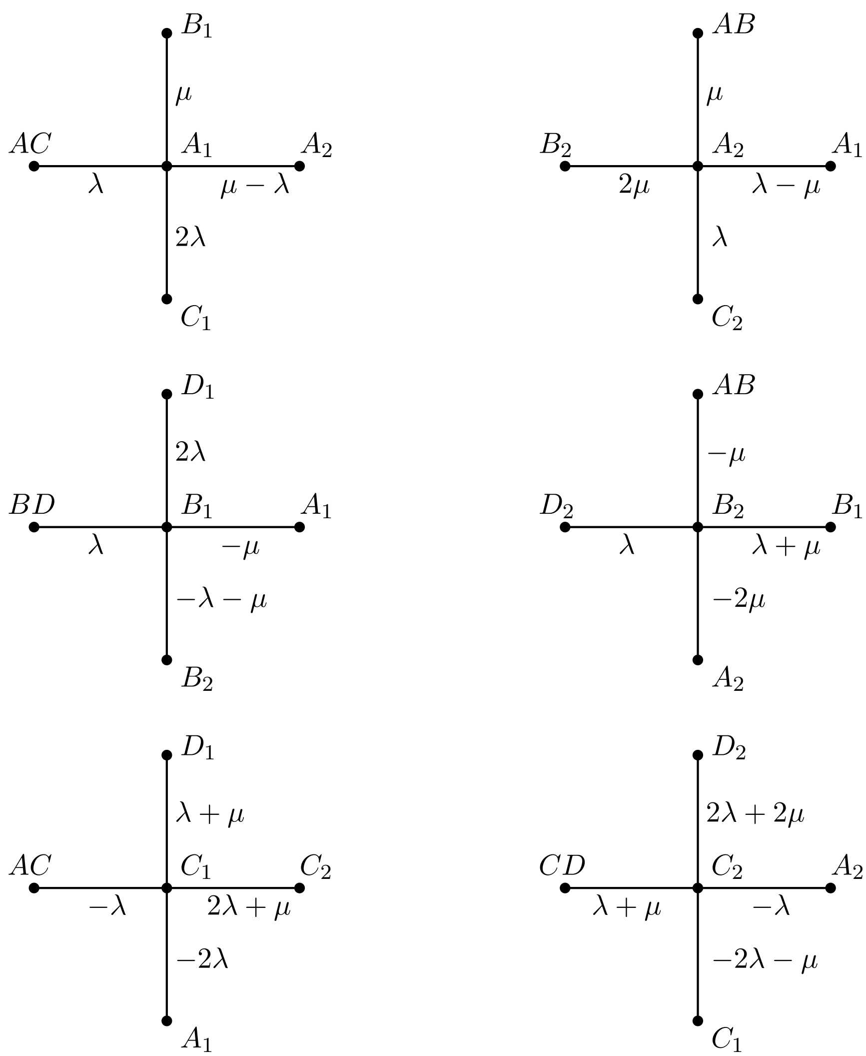
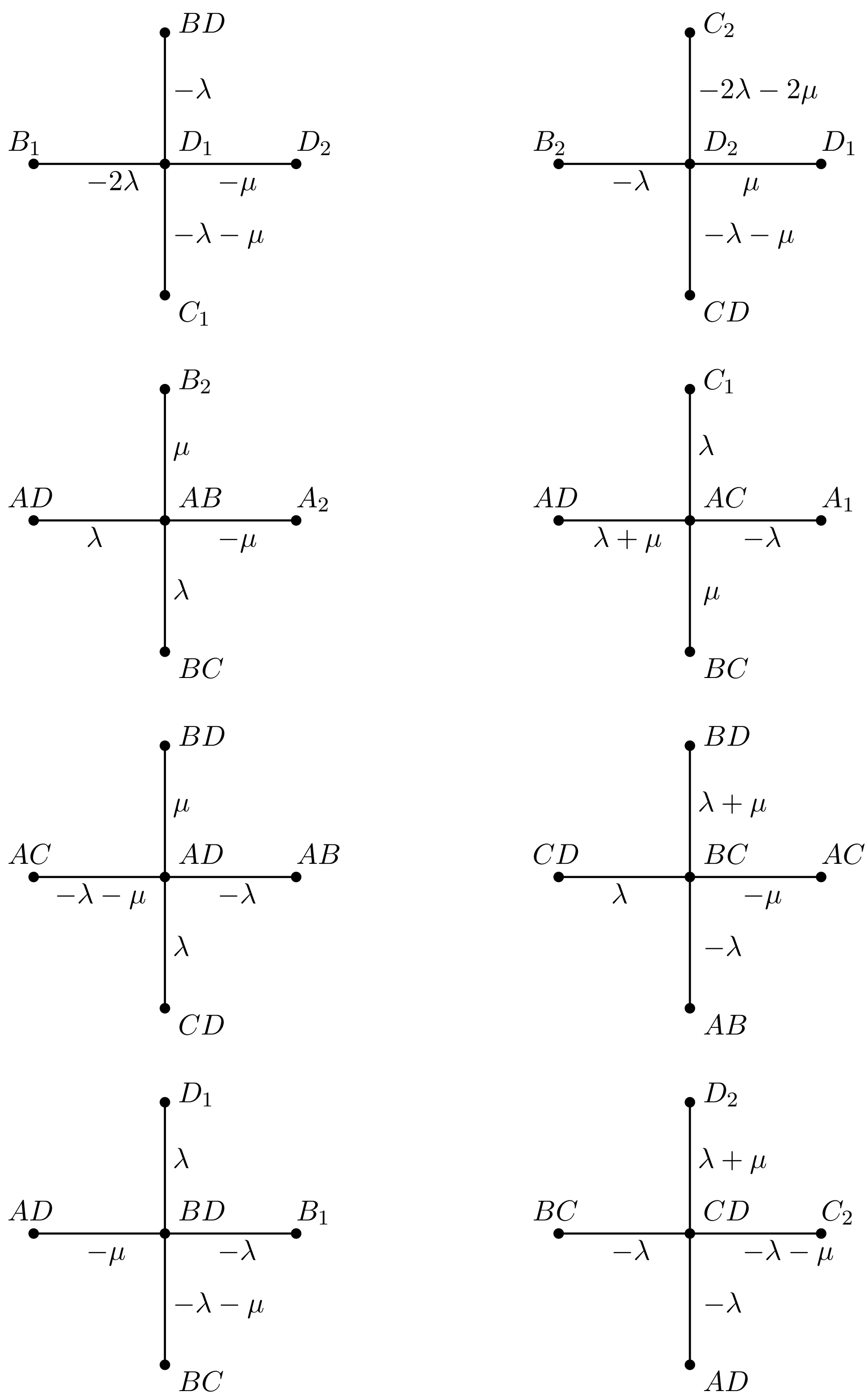


Figure 3.3: Weights At Fixed Points(cont.)



Chapter 4

Computation of Gromov-Witten Invariants

When a smooth projective variety admits a torus action, the moduli space of stable maps automatically inherits a torus action. The fixed points of the induced action correspond to invariant lines of the action on the variety. When the torus action on the variety has finitely many fixed points and the invariant lines connecting the fixed points are isolated, the connected components of fixed point loci on the moduli space can be determined and recorded by graphs with additional data. The edges of the graphs represent the non-contracted components of the nodal curves mapped to the invariant lines, the vertices represent the contracted components mapped to the fixed points of the torus action. Then the virtual localization technique is used to compute Gromov-Witten invariants. There were various cases where people succeeded in applying this approach. See [10, 7, 20], etc.

Our example of Hilbert scheme $F_1^{[2]}$ exhibits slightly different property as we realized before, which is that there are one-dimensional families of invariant lines for some curve degrees. The point of our strategy to overcome the problem is for some curve classes, the collection of the relevant invariant lines may differ. If for some curve classes, all the relevant invariant lines are isolated, we can obtain all the connected components of the moduli spaces in the usual way and then apply the localization formula to compute Gromov-Witten invariants corresponding to these curve classes.

4.1 Connected Components Analysis

In this section we follow the presentation of [10, 7, 20], etc. Suppose X is a smooth projective variety with a torus T-action. Then the moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(X,\beta)$ with $\beta \in H_2(X,\mathbb{Z})$ admits a T-action by composing maps from nodal curves to X with torus action. A fixed point of the moduli space under the torus action has the following properties:

(1) all marked points, nodes, contracted components and ramification points are mapped to

fixed points in X;

(2) non-contracted components are mapped onto invariant lines, which are $\mathbb{P}^1$'s, and are ramified only over two fixed points connecting the lines.

When we fix a curve degree β, the sum degree of the images of the maps in the moduli space on the irreducible components of the nodal curves has to be β. This restricts the collection of invariant lines appearing in the connected components of fixed point loci. If for some curve class β, all the invariant lines which may appear in the connected components are isolated, the connected components of the fixed point loci can be described as follows: to each fixed stable map $f : C \to X$, we associate a marked graph Γ in the following way: Γ has one vertex v for each connected component in the inverse image under f of the fixed point set in X, which is labeled with the name of that fixed point if the component is mapped to that point; Γ has one edge e for every non-contracted component, whose two vertices are labeled with two different fixed points and which is labeled with the degree d_e of the map from the component to its image line. Also we label each vertex v with a leg for each marked point which is mapped to its corresponding fixed point. We usually use numerals to denote the legs in graphs. Then all the stable maps with the same corresponding graph Γ form a connected component and all connected components are described by all such graphs.

Denote the valence $val(v)$ of a vertex v as the number of edges and legs attached to it. For each graph Γ, define

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \Gamma} \overline{\mathcal{M}}_{0,val(v)},$$

where we adopt the convention that

$$\overline{\mathcal{M}}_{0,1} = \overline{\mathcal{M}}_{0,2} = pt.$$

This is a DM-stack. There is a universal family of T-fixed stable maps to X,

$$\pi : \mathcal{C} \to \overline{\mathcal{M}}_\Gamma,$$

which induces a morphism $\gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{0,n}(X,\beta)$.

The automorphism group A of this family is filtered by an exact sequence

$$1 \to \prod_{e\in\Gamma} \mathbb{Z}/(d_e) \to A \to Aut(\Gamma) \to 1,$$

where $Aut(\Gamma)$ denotes the automorphism group of Γ. The induced morphism

$$\gamma/A : \overline{\mathcal{M}}_\Gamma/A \to \overline{\mathcal{M}}_{0,n}(X,\beta)$$

is a closed immersion of DM-stacks and is realized as a connected component of the moduli space under the torus action.

4.2 The Virtual Normal Bundle

In this section, we recall the standard computations in [10, 7, 20].

Over each connected component $\overline{\mathcal{M}}_\Gamma/A$ of fixed point loci, we describe the obstruction theory of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ restricted to it. From the second chapter, we know there is a canonical perfect obstruction theory on $\overline{\mathcal{M}}_{0,n}(X,\beta)$ defined by a two-term complex $E_0 \to E_1$, whose kernel T^1 is the tangent space of the moduli space and whose cokernel T^2 represents the obstruction theory for stable maps. They are related in the exact sequence

$$\begin{aligned} 0 \to Ext^0(\Omega_C(D), \mathcal{O}) \to H^0(C, f^*TX) \to T^1 \to Ext^1(\Omega_C(D), \mathcal{O}) \\ \to H^1(C, f^*X) \to T^2 \to 0, \end{aligned}$$

in which D represents the divisor of marked points on C. When restricted to the connected component $\overline{\mathcal{M}}_\Gamma/A$, the four terms other than the sheaves T^1 and T^2 form vector bundles as fibres. We call them as B_1, B_2, B_4 and B_5 respectively. Each vector bundle B_i decomposes as the direct sum of the fixed part B_i^f and the moving part B_i^m under the torus action. The moving parts inherit a natural T-action. From the above exact sequence, we have

$$e^{\mathbb{C}^*}(N_\Gamma^{vir}) = \frac{e^{\mathbb{C}^*}(B_2^m)e^{\mathbb{C}^*}(B_4^m)}{e^{\mathbb{C}^*}(B_1^m)e^{\mathbb{C}^*}(B_5^m)},$$

where N_Γ^{vir} stands for the virtual normal bundle of the connected component.

This is the denominator in the localization formula and has to be worked out to apply localization formula. In order to do that, we have to work out each term one by one. But first we do some preparatory work.

Lemma 4.2.1. *For any invariant line $\mathbb{P}^1$ connecting two fixed points p_1 and p_2, there exist a weight λ_1 at p_1 and a weight λ_2 at p_2 such that they differ by a negative sign.*

Proof. Without loss of generality, we can choose a coordinate system such that p_1 is the origin and p_2 is the infinity. So the action $T \times \mathbb{P}^1 \to \mathbb{P}^1$ takes the form $tx = b(t)x$ for any $t \in T$ and $x \in \mathbb{P}^1 - \infty$. It is obvious that $b : T \to T$ is a character, so it takes the form $b(t) = t^\lambda$ for some character λ. The induced action on $T_{p_1}\mathbb{P}^1$ has the weight λ and since $T_{p_1}\mathbb{P}^1$ is a subbundle in $T_{p_1}X$, the T-action on $T_{p_1}X$ has λ as a weight. If we reverse the role of p_1 and p_2, we get $-\lambda$ as a weight for $T_{p_2}X$, which finishes the proof. $\square$

We need some notations. A flag F is a pair (v, e), where v is a vertex and e is an edge to which v is attached. We adopt the convention $v(F) = v$, $e(F) = e$ and $i(F) = i(v)$, the marking of the nodal point in the vertex v. We denote $j(F)$ to be the marking of the other vertex of the edge e. The special weight at $p_{i(F)}$ corresponding to e determined in the previous lemma is denoted as λ_F.

$\mathcal{V}_{s,t}$ = the subset of vertices with s flags and t legs;

$\mathcal{V}_s$ = the subset of vertices with s flags or legs;

$\mathcal{F}_{s,t}$ = the subset of flags whose vertices are in $\mathcal{V}_{s,t}$;

$\mathcal{F}_s$ = the subset of flags whose vertices are in $\mathcal{V}_s$.

Lemma 4.2.2. *For a flag $F = (v, e)$, the T-action on $T_v\mathbb{P}^1$ induced by that on X has weight $\frac{\lambda_F}{d_e}$.*

Proof. As we did in the proof of Lemma 1, we can choose a coordinate system for $\mathbb{P}^1 \in X$ such that $p_{i(F)}$ is the origin and $p_{j(F)}$ is the infinity, and so the action $T \times \mathbb{P}^1 \to \mathbb{P}^1$ takes the form $tx = t^{\lambda_F}x$ for any $t \in T$, $x \in \mathbb{P}^1 - \infty$ and some character λ_F. We can also choose a coordinate system for $\mathbb{P}^1$

corresponding to e, so that the map f takes the form $y = f(z) = z^{d_e}$ for any $z \in \mathbb{P}^1 - \infty$. Now under the T-action on X, $ty = t^{\lambda_F} y$, so $tz = t^{\frac{\lambda_F}{d_e}} z$, and hence the T-action on $T_v\mathbb{P}^1$ has weight $\frac{\lambda_F}{d_e}$. $\square$

4.2.1 B_1^m

$B_1 = Ext^0(\Omega_C(D), \mathcal{O})$, also denoted Aut_∞^m, parameterizes infinitesimal automorphisms of the pointed nodal curve. It decomposes into the direct sum of those on each irreducible component of the nodal curve. Apparently, T acts on the contracted components trivially, so B_1^m is the direct sum of the moving parts on the non-contracted components, i.e.

$$B_1^m = \oplus_{edges} Aut_\infty^m(C_e).$$

Following [20], we divide the situation in two cases.

Case 1: $F \in \mathcal{F}_1$, and the other vertex is in $\mathcal{V}_{\geq 2}$. Treating this other vertex as ∞ in the non-contracted component, we have the automorphism φ takes the form $\varphi(x) = ax + b$, for $x \in P^1 - \infty$, and $0 \neq a, b \in C$. We also know that $tx = t^{\lambda_F} x$, so

$$t\varphi t^{-1}(x) = t\varphi(t^{-\frac{\lambda_F}{d_e}} x) = t(at^{-\frac{\lambda_F}{d_e}} x + b) = t^{\frac{\lambda_F}{d_e}}(at^{-\frac{\lambda_F}{d_e}} x + b) = ax + t^{\frac{\lambda_F}{d_e}} b.$$

This means that the T-action on the infinitesimal automorphisms of this non-contracted component has weight $\frac{\lambda_F}{d_e}$.

Case 2: Neither vertices is in $\mathcal{V}_1$. Then the automorphism φ takes the form $\varphi(x) = ax$, for $x \in P^1 - \infty$, and $0 \neq a \in C$. As above,

$$t\varphi t^{-1}(x) = t\varphi(t^{-\frac{\lambda_F}{d_e}} x) = t(at^{-\frac{\lambda_F}{d_e}} x) = t^{\frac{\lambda_F}{d_e}}(at^{-\frac{\lambda_F}{d_e}} x) = ax,$$

which means that the T-action on the infinitesimal automorphisms of this non-contracted component has weight 0. Putting together, we get

$$e(B_1^m) = \prod_{F \in \mathcal{F}_1} \frac{\lambda_F}{d_e}.$$

4.2.2 B_4^m

$B_4 = Ext^1(\Omega_C(D), \mathcal{O})$, also denoted $Def(C(D))$, represents the space of deformations of the pointed nodal curve. Again the torus action on the deformations within each contracted component is trivial and since non-contracted components are projective lines without nontrivial deformations, the moving part of B_4 is the direct sum of moving parts of smoothing nodes between non-contracted components and contracted components or between pairs of non-contracted components. A well-known fact is that smoothing a node is identified as a bundle with fibres the tensor product of the tangent spaces of the two components at the node.

Let $\mathcal{L}_F$ be the universal cotangent line bundle over $\overline{\mathcal{M}}_\Gamma$ at the nodal point corresponding to $F \in \mathcal{F}_{(2,0)} \cup \mathcal{F}_{\geq 3}$, and write $e_F = c_1(\mathcal{L}_F)$. When F does not belong to this union, we set $e_F = 1$. We still need to consider two different cases.

Case 1: $F \in \mathcal{F}_{\geq 3}$. Then at the vertex of F, a non-contracted component intersects with a contracted component. The weight of the T-action for this part is $\dfrac{\lambda_F}{d_e} - e_F$.

Case 2: $F \in \mathcal{F}_{(2,0)}$. Then at the vertex of F, two non-contracted components intersect. The weight of the T-action for this part is $\dfrac{\lambda_F}{d_e} + \dfrac{\lambda_G}{d_{e'}}$, where (G, e') is the other flag.

So,

$$e(B_4^m) = \prod_{F \in \mathcal{F}_{\geq 3}} (\frac{\lambda_F}{d_e} - e_F) \prod_{F \in \mathcal{F}_{(2,0)}} (\frac{\lambda_F}{d_e} + \frac{\lambda_G}{d_{e'}}).$$

4.2.3 $B_2^m - B_5^m$

B_2 and B_5 represents the spaces of deformations and obstructions of the maps. The normalization sequence resolving all of the nodes of C is

$$0 \to \mathcal{O}_C \to \bigoplus_{verts} \mathcal{O}_{C_v} \oplus \bigoplus_{edges} \mathcal{O}_{C_e} \to \bigoplus_{flags} \mathcal{O}_{x_F} \to 0,$$

where in the middle term, the vertices that the sum is over have valence at least 3. After being twisted by f^*TX, this sequence gives rise to an exact sequence of cohomology groups,

$$
\begin{aligned}
0 \to H^0(f^*TX) \to \bigoplus_{verts} H^0(C_v, f^*TX) \oplus \bigoplus_{edges} H^0(C_e, f^*TX) \to \bigoplus_{flags} T_{p_{i(F)}}X \\
\to H^1(f^*TX) \to \bigoplus_{verts} H^1(C_v, f^*TX) \oplus \bigoplus_{edges} H^1(C_e, f^*TX) \to 0,
\end{aligned}
$$

in which we use the fact that $H^1(\mathcal{O}_{x_F}, f^*TX) = 0$. Note that $H^0(C_v, f^*TX) = T_{p_{i(v)}}X$, because C_v is connected and f is constant on it. Also, since we only consider genus zero invariants, C_v is a genus zero stable curve for each vertex v. So $H^1(C_v, f^*TX) = 0$. So, we have

$$
\begin{aligned}
H^0 - H^1 = & \bigoplus_{verts} T_{p_{i(v)}}X - \bigoplus_{flags} T_{p_{i(F)}}X \\
& + \bigoplus_{edges} H^0(C_e, f^*TX) - \bigoplus_{edges} H^1(C_e, f^*TX)
\end{aligned}
$$

The weights on $\bigoplus_{verts} T_{p_{i(v)}}X$ and $\bigoplus_{flags} T_{p_{i(F)}}X$ are given as the assumption. Also with the weights at the two fixed points 0 and ∞ of the T-action on P^1, we can use localization formula in equivariant K-theory to compute the weights on the virtual bundle $H^0(C_e, f^*TX) - H^1(C_e, f^*TX)$. Thus the equivariant Euler class of the virtual bundle $B_2^m - B_5^m$ is determined. Putting all these together, we obtain the equivariant Euler class of the virtual normal bundle N_Γ^{vir}.

4.3 One-Point Gromov-Witten Invariants

Let's first concentrate on the study of moduli space of stable maps of degree $d\beta_1$ for a positive integer d. To get nontrivial Gromov-Witten invariants $< \alpha_1, \alpha_2, \cdots, \alpha_n >_{0,n,d\beta_1}$, for $\alpha_i \in A^*(F_a^{[2]})$, the cohomological degrees of α_i should add up to $n+1$, i.e.

$$\sum_{i=1}^{n} deg(\alpha_i) = n+1.$$

This happens only when one class has degree 2 and other classes all have degree 1. And by Divisor Axiom of Gromov-Witten invariants, when $deg(\alpha_n) = 1$,

$$< \alpha_1, \alpha_2, \cdots, \alpha_n >_{0,n,d\beta_1} = \int_{d\beta_1} \alpha_n < \alpha_1, \alpha_2, \cdots, \alpha_{n-1} >_{0,n-1,d\beta_1} .$$

So by induction, the computation of Gromov-Witten invariants for curve classes $d\beta_1$ is reduced to that corresponding to a single class of degree 2, i.e. $< \alpha >_{0,1,d\beta_1}$, for $\alpha \in A^2(F_a^{[2]})$. From [13], we have

Theorem 4.3.1. *(i)* $< \beta_j >_{0,d\beta_1} = 0$ *for* $j = 6, 7, 8, 9$;

(ii) $< \beta_4 >_{0,d\beta_1} = -\frac{2}{d}, < \beta_5 >_{0,d\beta_1} = -\frac{4}{d}$.

Proof. (i) is clear from [13].

(ii) $< \beta_4 >_{0,d\beta_1} = 2(K_X \cdot S_0)/d = 2(-2S_0 - 3f) \cdot S_0/d = -\frac{2}{d}$,

$< \beta_5 >_{0,d\beta_1} = 2(K_X \cdot f)/d = 2(-2S_0 - 3f) \cdot f/d = -\frac{4}{d}$. □

From now on, we assume $a = 1$, and we write F for F_1. We know $A_1(F^{[2]})$ is freely generated by $\beta_1, \beta_2 - \beta_1$ and $\beta_3 - \beta_1$ and the invariant lines in Figure 3.2 can all be expressed as linear combinations in these generators with nonnegative coefficients. With the point of view of the virtual localization, GW-invariants of any number of points of any curve class β vanish except that

$$\beta = d\beta_1 + d_2(\beta_2 - \beta_1) + d_3(\beta_3 - \beta_1),$$

for some non-negative integers d, d_2, d_3.

With the above theorem, we assume that d_2, d_3 are not simultaneously zero. To compute the one-pointed GW-invariants, first the virtual dimension of $\overline{\mathcal{M}}_{0,1}(F^{[2]}, \beta)$ is

$$virdim\overline{\mathcal{M}}_{0,1}(F^{[2]}, \beta) = d_2 + 2d_3 + 2.$$

For the dimensional reason, we only need to consider $(d_2, d_3) = (1, 0), (2, 0)$ or $(0, 1)$ to get nonzero 1-point invariants. Our strategy for computations is that we choose a suitable cycle to represent the cohomology class so that there are only finitely many connected components of a specific curve class intersecting the cycle. Only these connected components have non-trivial contributions in the localization formula. Each connected component shows up as a tree of invariant lines with markings described by a graph. In the localization formula, the equivariant Euler classes of their virtual normal bundles have been worked out; the restriction of the cohomology class to the connected components can be decided by applying Corollary 2.4.1. In particular, if no tree of invariant lines of the required degree intersects the cycle, then the invariant vanishes. This prompts the idea that we purposely choose some representative of a cohomological class so that either it stays away from any such tree or intersects with as few such trees as possible. In this way, the computation with the localization formula is simplified.

For the pairs $(2,0)$ and $(0,1)$, the virtual dimension of the moduli space is equal to 4, so the insertion of nonzero Gromov-Witten invariants must be a point class.

Proposition 4.3.1. *For all curve class β,*

$$< pt >_{0,1,\beta} = 0,$$

except that $< pt >_{0,1,\beta_3} = 2$.

Proof. We first remark, in this proof and in the proofs of propositions throughout this chapter, we constantly refer to Figure 3.2 for configuration of fixed points and invariant lines and Figure 3.3 for relevant weights at fixed points.

For the first pair $(2,0)$, we take the point BD for the point class. Then any tree of invariant lines passing through BD has to contain β_3 from Figure 3.2, which is not allowed in $(2,0)$. So the localization formula expansion does not have any nonzero term in it. So $< pt >_{0,1,\beta} = 0$ in this case.

For the second pair $(0,1)$, we take the point AC for the point class. When $d \neq 1$, it is away from any tree of invariant lines of sum degree β. So $< pt >_{0,1,\beta} = 0$ in this case.

Now assume $d = 1$, i.e. $\beta = \beta_3$. Then there are two nonzero terms in the localization formula from connected components described by the following graphs:

where here and in the following, the boldface points mean where the marked points are mapped to and the numbers above line segments mean the degrees of the lines.

Now we determine the equivariant Euler classes of their virtual normal bundles. For Γ_1, $e^{\mathbb{C}^*}(B_1^m) = -\mu, e^{\mathbb{C}^*}(B_4^m) = 1$.

To compute $\dfrac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)}$, we use the localization formula to $f^*TF^{[2]}$ in equivariant topological K-theory. This technique has been used previously in a similar context by [6]. That is

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{\lambda+\mu}}{1-t^{-\mu}} + \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1-t^{\mu}} \\ &= 1 + t^\lambda + t^{-\lambda} + t^{-\mu} + t^\mu + t^{\lambda+\mu},\end{aligned}$$

then,

$$\frac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)} = \lambda^2\mu^2(\lambda+\mu).$$

Therefore,

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = \frac{e^{\mathbb{C}^*}(B_2^m)e^{\mathbb{C}^*}(B_4^m)}{e^{\mathbb{C}^*}(B_1^m)e^{\mathbb{C}^*}(B_5^m)} = -\lambda^2\mu(\lambda+\mu).$$

For Γ_2, $e^{\mathbb{C}^*}(B_1^m) = -\lambda-\mu, e^{\mathbb{C}^*}(B_4^m) = 1$. Again by the localization formula in equivariant topological K-theory,

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{-\lambda-\mu}}{1-t^{\lambda+\mu}} + \frac{t^\lambda + t^{\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1-t^{-\lambda-\mu}} \\ &= 1 + t^\lambda + t^{-\lambda} + t^\mu + t^{\lambda+\mu} + t^{-\lambda-\mu},\end{aligned}$$

so,

$$\frac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)} = \lambda^2\mu(\lambda+\mu)^2, \qquad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \frac{\lambda^2\mu(\lambda+\mu)^2}{-\lambda-\mu} = -\lambda^2\mu(\lambda+\mu).$$

Putting these term in the localization formula, we get

$$< pt >_{0,1,\beta_3} = \frac{-\lambda^2\mu(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)} + \frac{-\lambda^2\mu(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)} = 1 + 1 = 2,$$

where the numerators by Corollary 2.4.1 are equivariant point class restricted to the point AC, which is the product of all the weights at AC. $\square$

For the pair $(1,0)$, the virtual dimension of the moduli space is equal to 3, so we need to feed a cohomology class of degree 3 or homological class of degree 1 to get nonzero GW-invariants.

Proposition 4.3.2. *For $\beta = d\beta_1 + (\beta_2 - \beta_1)$,*

(i) $< \beta_1 >_{0,1,\beta} = 0$, for any d;

(ii) $< \beta_2 >_{0,1,\beta} = 0$, for any $d \neq 1$; $\quad -1$, for $d = 1$;

(iii) $< \beta_3 >_{0,1,\beta} = 0$, for any $d \neq 1$; $\quad 1$, for $d = 1$.

Proof. (i) Let's take the invariant line between B_1 and B_2 to be the representative of β_1. Since any tree of invariant lines touching this representative has to contain β_3, which is not allowed, we have $< \beta_1 >_{0,1,\beta} = 0$ for any $d \geq 0$.

(ii) For β_2, we take the invariant line between AB and BC as a representative. When $d \neq 1$, for the same reason, this invariant line does not intersect any tree of invariant lines of sum class β, so the GW-invariants are equal to zero. When $d = 1$, $\beta = \beta_2$. We have two nonzero terms in the localization formula from connected components described by the following graphs:

For Γ_1, $e^{\mathbb{C}^*}(B_1^m) = -\lambda, e^{\mathbb{C}^*}(B_4^m) = 1$.

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^\lambda + t^{-\mu}}{1-t^{-\lambda}} + \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1-t^\lambda} \\ &= 2t^\lambda + t^{-\lambda} + t^{-\mu} + 1,\end{aligned}$$

so,

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = \frac{\lambda^3\mu}{-\lambda} = -\lambda^2\mu.$$

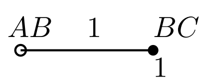
For Γ_2, $e^{\mathbb{C}^*}(B_1^m) = \lambda, e^{\mathbb{C}^*}(B_4^m) = 1$. Also, $\chi(f^*TF_a^{[2]})$ is the same as for Γ_1. So

$$e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \lambda^2\mu.$$

Using localization formula, we have

$$< \beta_2 >_{0,1,\beta_2} = \frac{-\lambda\mu^2}{-\lambda^2\mu} + \frac{-\lambda\mu(\lambda+\mu)}{\lambda^2\mu} = -1.$$

(iii) Finally, we come to $< \beta_3 >_{0,1,\beta}$. We take the invariant line from BC to BD to be the representative cycle of β_3. Again, when $d \neq 1$, this invariant line does not intersect any tree of invariant lines of sum class β, so the GW-invariants are equal to zero. When $d = 1$, only one component contributes a nonzero term, which is described by the graph:



Γ

where the marked point is mapped to BC. Now $e^{\mathbb{C}^*}(B_1^m) = \lambda, e^{\mathbb{C}^*}(B_4^m) = 1$. $\chi(f^*TF_a^{[2]})$ is equal to the corresponding term for Γ_1 or Γ_2 above. So,

$$e^{\mathbb{C}^*}(N_{\Gamma}^{vir}) = \lambda^2\mu.$$

Using localization formula, we have

$$< \beta_3 >_{0,1,\beta_2} = \frac{\lambda^2 \mu}{\lambda^2 \mu} = 1.$$

$\square$

Up to this point, we have computed all the 1-pointed Gromov-Witten invariants of $F^{[2]}$ for all curve classes.

4.4 Two-Point Gromov-Witten Invariants

When $n = 2$,

$$virdim\overline{\mathcal{M}}_{0,2}(F^{[2]}, \beta) = d_2 + 2d_3 + 3,$$

for $\beta = d\beta_1 + d_2(\beta_2 - \beta_1) + d_3(\beta_3 - \beta_1)$. To get nonzero invariants, we must have $d_2 + 2d_3 \leq 5$. The complete list of these pairs of (d_2, d_3) are $(5,0), (4,0), (3,0), (2,0), (1,0), (3,1), (2,1), (1,1), (0,1), (1,2)$ and $(0,2)$, in other word, we have

Proposition 4.4.1. *For $\beta = d\beta_1 + d_1(\beta_2 - \beta_1) + d_2(\beta_3 - \beta_1)$, where $(d_2, d_3) \neq (5,0), (4,0), (3,0),$ $(2,0), (1,0), (3,1), (2,1), (1,1), (0,1), (1,2), (0,2)$,*

$$< \alpha, \beta >_{0,2,\beta} = 0,$$

for any $\alpha, \beta \in H^*(F^{[2]}, \mathbb{Q})$.

In the following we shall treat different cases one by one. The strategy is almost the same as for computing one-point invariants: if we can choose a cycle representing one cohomology class which never intersects any tree of invariant lines of the required degree or if we can choose the representative cycles for the two classes which do not intersect any tree of invariant lines of the required degree simultaneously, the Gromov-Witten invariant in question must vanish. Generally, we keep the freedom for choosing such representatives so that there are as small numbers of such trees as possible with nonempty intersection with the chosen representatives.

Especially, when one insertion is a point class, this can be treated more readily. We are in such situation for the pairs $(5,0),(3,1),(1,2),(4,0),(2,1)$ and $(0,2)$. In the former three pairs, the virtual dimension of the moduli space is equal to 8, so the degree decomposition of the two insertions must be $4+4$. In the latter three pairs, the virtual dimension is 7, so the degree decomposition of the two insertions must be $3+4$.

Proposition 4.4.2. *For* $\beta = d\beta_1 + 5(\beta_2 - \beta_1), d\beta_1 + 3(\beta_2 - \beta_1) + (\beta_3 - \beta_1)$,

$$< pt, pt >_{0,2,\beta} = 0$$

for any d.

Proof. For $\beta = d\beta_1 + 5(\beta_2 - \beta_1)$, we choose the point BD to represent one point class. Then BD does not lie in any tree of invariant lines of sum degree β, so $< pt, pt >_{0,2,\beta} = 0$ in this case.

For $\beta = d\beta_1 + 3(\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, we choose the point BD to represent one point class and the point AC to represent another point class. Then any tree of invariant lines of sum degree β does not pass through both of the points simultaneously, so $< pt, pt >_{0,2,\beta} = 0$ in this case. □

Note for $\beta = d\beta_1 + (\beta_2 - \beta_1) + 2(\beta_3 - \beta_1), < pt, pt >_{0,2,\beta}$ is not computable by this method since no matter how we choose the representatives of the point class, the existence of one-dimensional families of invariant lines makes it impossible for us to only have isolated trees of invariant lines of the required curve class through these representatives.

Now for the latter three pairs, we have

Proposition 4.4.3. *For* $\beta = d\beta_1 + 4(\beta_2 - \beta_1), d\beta_1 + 2(\beta_3 - \beta_1)$ *and* $d\beta_1 + 2(\beta_2 - \beta_1) + (\beta_3 - \beta_1)$,

$$< pt, \beta_i >_{0,2,\beta} = 0,$$

for $i = 1, 2, 3$ *and any* d.

Proof. First for $\beta = d\beta_1 + 4(\beta_2 - \beta_1)$, if we take the point BD for the point class, then any tree of invariant lines of the designated degree cannot pass through this point, so $< pt, \beta_i >_{0,2,\beta} = 0$ for $i = 1, 2, 3$.

Then for $\beta = d\beta_1 + 2(\beta_3 - \beta_1)$, we choose the point D_1 for the representative of the point class, the invariant line between A_1 and A_2 for β_1, the invariant line between AB and BC for β_2, the invariant line between BC and AC for β_3. Then we see $< pt, \beta_i >_{0,2,\beta} = 0$, for $i = 1, 2, 3$.

Finally assume $\beta = d\beta_1 + 2(\beta_2 - \beta_1) + (\beta_3 - \beta_1)$. If we take the point BD for the point class and keep the representative for β_1 as above, then we see $< pt, \beta_1 >_{0,2,\beta} = 0$; if we take the point D_1 for the point class and keep the representative for β_2 as above, then we see $< pt, \beta_2 >_{0,2,\beta} = 0$.

To consider $< pt, \beta_3 >_{0,2,\beta}$, we take D_2 for the point class and the line between B_2 and AB for the representative of $\beta_3 - \beta_1$, then we see there isn't any tree of invariant lines of this degree connecting two cycles. So $< pt, \beta_3 - \beta_1 >_{0,2,\beta} = 0$. But $< pt, \beta_1 >_{0,2,\beta} = 0$ from above, so we have $< pt, \beta_3 >_{0,2,\beta} = 0$. □

For the pairs $(d_2, d_3) = (3, 0), (1, 1)$, the virtual dimension of the moduli space is equal to 6. Then the degree decomposition of the two insertions is either $2 + 4$ or $3 + 3$. The first type is dealt with in the following

Proposition 4.4.4. *(i) For* $\beta = d\beta_1 + 3(\beta_2 - \beta_1)$,

$$< pt, \beta_i >_{0,2,\beta} = 0,$$

for $i = 4, 5, 6, 7, 8, 9$ *and any* d*;*

(ii) For $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, *we have*

$$< pt, \beta_4 >_{0,2,\beta} = < pt, \beta_6 >_{0,2,\beta} = < pt, \beta_8 >_{0,2,\beta} = 0,$$

for any d*. Also, for* $d \neq 2$,

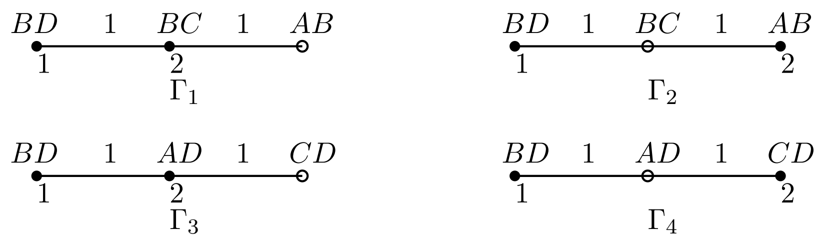
$$< pt, \beta_9 >_{0,2,\beta} = 0,$$

but for $d = 2$, $< pt, \beta_9 >_{0,2,\beta} = 2$.

Proof. (i) We take the point BD for the representative for point class and the standard representatives for β_i listed in §3.2, where f_0 is assigned to the undesignated f's and the point A assigned to pt's in those expressions. Then we see that any tree of invariant lines has to contain β_3, which is excluded by the given curve class, so $< pt, \beta_i >_{0,2,\beta} = 0$ for $i = 4, \cdots, 9$.

(ii) For $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, if we take the point BD for the point class and the standard representative for β_4, then we see that $< pt, \beta_4 >_{0,2,\beta} = 0$ for any d.

Now we compute $< pt, \beta_6 >_{0,2,\beta}$. We choose the point BD to represent the point class and the standard representative for β_6. Then we see that $< pt, \beta_6 >_{0,2,\beta} = 0$ if $d \neq 2$. When $d = 2$, there are nonzero terms from the fixed point loci described by the graphs:



For Γ_1, $e^{\mathbb{C}^*}(B_1^m) = \lambda, e^{\mathbb{C}^*}(B_4^m) = -\lambda(\lambda + \mu)$. To compute $\dfrac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)}$, let's use e_1 to denote the invariant line from BC to BD, e_2 to denote the invariant line from AB to BC. Then by localization,

$$\chi(f^*TF^{[2]}|_{e_1}) = \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1 - t^{-\lambda-\mu}} + \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{-\lambda-\mu}}{1 - t^{\lambda+\mu}}$$
$$= t^\lambda + t^{-\lambda} + t^{-\mu} + t^{\lambda+\mu} + t^{-\lambda-\mu} + 1.$$

Also $\chi(f^*TF^{[2]}|_{e_2}) = 2t^\lambda + t^{-\lambda} + t^{-\mu} + 1$ from before. Then applying normalization sequence,

$$\chi(f^*TF^{[2]}) = \chi(f^*TF^{[2]}|_{e_1}) + \chi(f^*TF^{[2]}|_{e_2}) - TF^{[2]}|_{BC}$$
$$= t^{-\lambda-\mu} + 2t^\lambda + t^{-\lambda} + t^{-\mu} + 2.$$

From this we have

$$\frac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)} = -\lambda^3\mu(\lambda + \mu),$$

and hence

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = \lambda^3\mu(\lambda+\mu)^2.$$

For Γ_2, $e^{\mathbb{C}^*}(B_1^m) = 1, e^{\mathbb{C}^*}(B_4^m) = -\lambda + (\lambda+\mu) = \mu, \dfrac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)}$ is the same as for Γ_1, so

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = -\lambda^3\mu^2(\lambda+\mu).$$

For Γ_3, $e^{\mathbb{C}^*}(B_1^m) = -\lambda, e^{\mathbb{C}^*}(B_4^m) = \lambda\mu$. We use e_1 to denote the invariant line from AD to BD, e_2 to denote the invariant line from AD to CD. Then,

$$\begin{aligned}
\chi(f^*TF^{[2]}|_{e_1}) &= \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{-\lambda-\mu}}{1-t^{-\mu}} + \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{-\lambda-\mu}}{1-t^{\mu}} \\
&= t^\lambda + t^{-\lambda} + t^\mu + t^{-\mu} + t^{-\lambda-\mu} + 1, \\
\chi(f^*TF^{[2]}|_{e_2}) &= \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{-\lambda-\mu}}{1-t^{-\lambda}} + \frac{t^{-\lambda} + t^{\lambda+\mu} + t^{-\lambda} + t^{-\lambda-\mu}}{1-t^{\lambda}} \\
&= t^\lambda + 2t^{-\lambda} + t^{-\lambda-\mu} + 1,
\end{aligned}$$

$$\begin{aligned}
\chi(f^*TF^{[2]}) &= \chi(f^*TF^{[2]}|_{e_1}) + \chi(f^*TF^{[2]}|_{e_2}) - TF^{[2]}|_{AD} \\
&= t^{-\mu} + t^{-\lambda-\mu} + t^\lambda + 2t^{-\lambda} + 2.
\end{aligned}$$

So

$$\frac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)} = \lambda^3\mu(\lambda+\mu), e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = -\lambda^3\mu^2(\lambda+\mu).$$

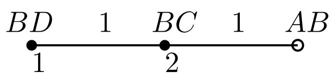
For Γ_4, $e^{\mathbb{C}^*}(B_1^m) = 1, e^{\mathbb{C}^*}(B_4^m) = \lambda+\mu, \dfrac{e^{\mathbb{C}^*}(B_2^m)}{e^{\mathbb{C}^*}(B_5^m)}$ is the same as for Γ_3, so

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4}) = \lambda^3\mu(\lambda+\mu)^2.$$

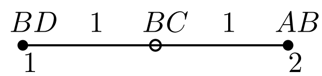
Using localization formula, we have

$$
\begin{aligned}
< pt, \beta_6 >_{0,2,\beta_2+\beta_3} =& \frac{\lambda^2\mu(\lambda+\mu)\mu(\lambda+\mu)}{\lambda^3\mu(\lambda+\mu)^2} + \frac{\lambda^2\mu(\lambda+\mu)\mu^2}{-\lambda^3\mu^2(\lambda+\mu)} \\
&+ \frac{\lambda^2\mu(\lambda+\mu)\mu(\lambda+\mu)}{-\lambda^3\mu^2(\lambda+\mu)} + \frac{\lambda^2\mu(\lambda+\mu)(\lambda+\mu)^2}{\lambda^3\mu(\lambda+\mu)^2} \\
=& 0.
\end{aligned}
$$

To compute $< pt, \beta_8 >_{0,2,\beta}$, we take the point BD for point class and the standard representative for β_8, where f is taken to be f_0. Then $< pt, \beta_8 >_{0,2,\beta} = 0$ when $d \neq 2$. When $d = 2$, there are two nontrivial connected components contributing to localization described by the graphs:



Γ_1



Γ_2

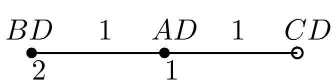
The equivariant Euler classes of the normal bundles are the same as those of Γ_1 and Γ_2 for $< pt, \beta_6 >_{0,2,\beta_2+\beta_3}$, or

$$
e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = \lambda^3\mu(\lambda+\mu)^2, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = -\lambda^3\mu^2(\lambda+\mu).
$$

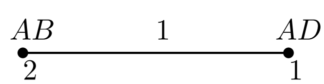
So

$$
\begin{aligned}
< pt, \beta_8 >_{0,2,\beta_2+\beta_3} =& \frac{-\lambda^2\mu(\lambda+\mu)\lambda(\lambda+\mu)}{\lambda^3\mu(\lambda+\mu)^2} + \frac{-\lambda^2\mu(\lambda+\mu)\lambda\mu}{-\lambda^3\mu^2(\lambda+\mu)} \\
=& 0.
\end{aligned}
$$

If we take the point AD for point class and the point B in the standard representative for β_9, then we see $< pt, \beta_9 >_{0,2,\beta} = 0$ when $d \neq 2$. When $d = 2$, there are nonzero terms in the localization formula from the connected components described by the graphs:



Γ_1



Γ_2

Γ_1 is the same as for Γ_3 for $< pt, \beta_6 >_{0,2,\beta_2+\beta_3}$ and its equivariant normal bundle has been worked out, i.e. $e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = -\lambda^3\mu^2(\lambda+\mu)$.

For Γ_2, $e^{\mathbb{C}^*}(B_1^m) = 1, e^{\mathbb{C}^*}(B_4^m) = 1$. By equivariant K-theoretic localization formula,

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^\lambda + t^{-\mu}}{1-t^{-\lambda}} + \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{-\lambda-\mu}}{1-t^\lambda} \\ &= 2t^\lambda + t^{-\lambda} + t^{-\mu} + t^\mu + t^{-\lambda-\mu} + 1.\end{aligned}$$

From this we have

$$e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = -\lambda^3\mu^2(\lambda+\mu).$$

So by localization formula,

$$< pt, \beta_9 >_{0,2,\beta_2+\beta_3} = \frac{-\lambda^2\mu(\lambda+\mu)\lambda\mu}{-\lambda^3\mu^2(\lambda+\mu)} + \frac{-\lambda^2\mu(\lambda+\mu)\lambda\mu}{-\lambda^3\mu^2(\lambda+\mu)} = 2.$$

□

Here the invariants $< pt, \beta_5 >_{0,2,\beta}, < pt, \beta_7 >_{0,2,\beta}$, when $\beta = d\beta_1 + (\beta_2-\beta_1) + (\beta_3-\beta_1)$ are not treated because the method does not apply. We will take on these in the last chapter.

For the pairs $(d_2; d_3) = (3; 0); (1; 1)$, the second type decomposition $3+3$ is dealt with in the following

Proposition 4.4.5. *(i) For* $\beta = d\beta_1 + 3(\beta_2 - \beta_1)$,

$$< \beta_i, \beta_j >_{0,2,\beta} = 0,$$

for $i, j = 1, 2, 3$ *and any* d*;*

(ii) For $\beta = d\beta_1 + (\beta_2-\beta_1) + (\beta_3-\beta_1)$,

$$< \beta_1, \beta_2 >_{0,2,\beta} = 0,$$

for any d and

$$<\beta_2,\beta_2>_{0,2,\beta}=<\beta_2,\beta_3>_{0,2,\beta}=0,$$

for any $d \neq 2$ *but when* $d=2$, $<\beta_2,\beta_2>_{0,2,\beta}=-1, <\beta_2,\beta_3>_{0,2,\beta}=1$.

Proof. (i) Let's first consider the case $\beta = d\beta_1 + 3(\beta_2-\beta_1)$. If we take the invariant line between B_1 and B_2 for the representative of β_1, it stays away from any tree of invariant lines of sum degree β. So $<\beta_1,\beta_j>_{0,2,\beta}=0$ for $j=1,2,3$.

If we take the invariant line between AB and BC for one representative of β_2 and the invariant line between AD and CD for another representative of β_2, then the pair of representatives do not touch any tree of invariant lines of sum degree β simultaneously. So we have $<\beta_2,\beta_2>_{0,2,\beta}=0$.

If we still take the invariant line between AB and BC for the representative of β_2 and the invariant line between AD and BD for the representative of β_3, then for the same reason, we have $<\beta_2,\beta_3>_{0,2,\beta}=0$.

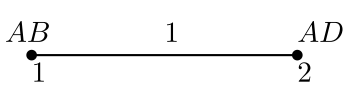
If we take the invariant line between BC and BD for one representative of β_3 and the invariant line between AC and AD for another representative of β_3, then the pair of representatives does not intersect any tree of invariant lines of sum degree β simultaneously. So we have $<\beta_3,\beta_3>_{0,2,\beta}=0$.

(ii) Then we consider the case when $\beta = d\beta_1 + (\beta_2-\beta_1)+(\beta_3-\beta_1)$.

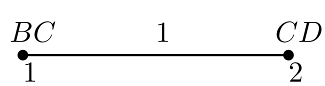
If we take the invariant line between B_1 and B_2 for the representative of β_1, the invariant line between AD and CD for the representative of β_2, then we see that $<\beta_1,\beta_2>_{0,2,\beta}=0$.

When $d\neq 2$, we take the invariant lines between AB and BC and between AD and CD for two representatives of β_2 and the invariant line between AD and BD for the representative of β_3, then we see that $<\beta_2,\beta_2>_{0,2,\beta}=<\beta_2,\beta_3>_{0,2,\beta}=0$.

Let's assume $d=2$ in the following. We first compute $<\beta_2,\beta_2>_{0,2,\beta_2+\beta_3}$. The nonzero terms in the localization formula are contributed from the fixed components described by the graphs:



Γ_1



Γ_2

For Γ_1, $e^{\mathbb{C}^*}(B_1^m) = 1, e^{\mathbb{C}^*}(B_4^m) = 1$, and

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^\mu + t^\lambda + t^{-\mu}}{1 - t^{-\lambda}} + \frac{t^\lambda + t^\mu + t^{-\lambda} + t^{-\lambda-\mu}}{1 - t^\lambda} \\ &= 2t^\lambda + t^{-\lambda} + t^{-\mu} + t^\mu + t^{-\lambda-\mu} + 1.\end{aligned}$$

So

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = -\lambda^3\mu^2(\lambda + \mu).$$

For Γ_2, again $e^{\mathbb{C}^*}(B_1^m) = 1, e^{\mathbb{C}^*}(B_4^m) = 1$.

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^\lambda + t^{-\mu} + t^{-\lambda} + t^{\lambda+\mu}}{1 - t^{-\lambda}} + \frac{t^{-\lambda} + t^{\lambda+\mu} + t^{-\lambda} + t^{-\lambda-\mu}}{1 - t^\lambda} \\ &= t^\lambda + 2t^{-\lambda} + t^{-\mu} + t^{-\lambda-\mu} + t^{\lambda+\mu} + 1.\end{aligned}$$

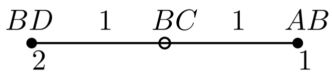
From this we have

$$e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \lambda^3\mu(\lambda + \mu)^2.$$

Using localization,

$$\begin{aligned}< \beta_2, \beta_2 >_{0,2,\beta_2+\beta_3} =& \frac{-\lambda\mu^2\lambda\mu(\lambda+\mu)}{-\lambda^3\mu^2(\lambda+\mu)} + \frac{-\lambda\mu(\lambda+\mu)\lambda(\lambda+\mu)^2}{\lambda^3\mu(\lambda+\mu)^2} \\ =& -1.\end{aligned}$$

To compute $< \beta_2, \beta_3 >_{0,2,\beta_2+\beta_3}$, we take the invariant lines between AB and BC for β_2, the invariant lines between AD and BD for β_3. Three nonzero terms appear in the localization formula described by the graphs:



Γ_3

The Euler classes of the virtual normal bundles of these components have been worked out above:

$$
\begin{aligned}
e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) &= -\lambda^3\mu^2(\lambda+\mu), \\
e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) &= \lambda^3\mu(\lambda+\mu)^2, \\
e^{\mathbb{C}^*}(N_{\Gamma_3}^{vir}) &= -\lambda^3\mu^2(\lambda+\mu).
\end{aligned}
$$

So

$$
\begin{aligned}
<\beta_2, \beta_3>_{0,2,\beta_2+\beta_3} =& \frac{-\lambda\mu^2\lambda^2(\lambda+\mu)}{-\lambda^3\mu^2(\lambda+\mu)} + \frac{-\lambda\mu(\lambda+\mu)\lambda^2(\lambda+\mu)}{\lambda^3\mu(\lambda+\mu)^2} + \frac{\lambda^2(\lambda+\mu)(-\lambda)\mu^2}{-\lambda^3\mu^2(\lambda+\mu)} \\
=& 1.
\end{aligned}
$$

□

For $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1), <\beta_1, \beta_1>, <\beta_1, \beta_3>$ and $<\beta_3, \beta_3>$ are not computable by this method. We'll take on this problem in the next chapter.

For the pairs $(d_2, d_3) = (2, 0), (0, 1)$, the virtual dimension of the moduli space is equal to 5, which can be decomposed as either $1 + 4$ or $2 + 3$. When the degree decomposition is $1 + 4$, we must have one insertion to be a point class.

Proposition 4.4.6. *(i) For* $\beta = d\beta_1 + 2(\beta_2 - \beta_1)$,

$$<pt, \beta_j>_{0,2,\beta} = 0,$$

for $j = 10, 11, 12$ *and any* d*;*

(ii) For $\beta = d\beta_1 + (\beta_3 - \beta_1)$,

$$<pt, \beta_j>_{0,2,\beta} = 0,$$

for $j = 10, 11, 12$ *and any* $d \neq 1$*; when* $d = 1$,

$$< pt, \beta_{10} >_{0,2,\beta_3} = 0, \;\; < pt, \beta_{11} >_{0,2,\beta_3} = 2, \;\; < pt, \beta_{12} >_{0,2,\beta_3} = 0.$$

Proof. By Axiom of Divisors,

$$< pt, \beta_j >_{0,2,\beta} = \int_\beta \beta_j < pt >_{0,1,\beta} .$$

So when $\beta = d\beta_1 + 2(\beta_2 - \beta_1)$ for any d and when $\beta = d\beta_1 + (\beta_3 - \beta_1)$ for any $d \neq 1$, $< pt, \beta_j >_{0,2,\beta} = 0$ since $< pt >_{0,1,\beta} = 0$ by Proposition 4.3.1

When $\beta = \beta_3$,

$$< pt, \beta_{10} >_{0,2,\beta_3} = \int_{\beta_3} \beta_{10} < pt >_{0,1,\beta_3} = 0,$$

because the intersection product of β_3 and β_{10} is 0,

$$< pt, \beta_{11} >_{0,2,\beta_3} = \int_{\beta_3} \beta_{11} < pt >_{0,1,\beta_3} = 1 \cdot 2 = 2,$$
$$< pt, \beta_{12} >_{0,2,\beta_3} = \int_{\beta_3} \beta_{12} < pt >_{0,1,\beta_3} = 0,$$

because the intersection product of β_3 with β_{11} and β_{12} are 1 and 0 respectively. □

The other degree decomposition is $2 + 3$.

Proposition 4.4.7. *(i) For* $\beta = d\beta_1 + 2(\beta_2 - \beta_1)$,

$$< \beta_i, \beta_j >_{0,2,\beta} = 0,$$

for $i = 1, 2, 3, j = 4, \cdots, 9$ *and any* d*;*

(ii) For $\beta = d\beta_1 + (\beta_3 - \beta_1)$,

$< \beta_1, \beta_j >_{0,2,\beta} = 0$ *for* $j = 5, 7, 8, 9$ *and all* d*;*

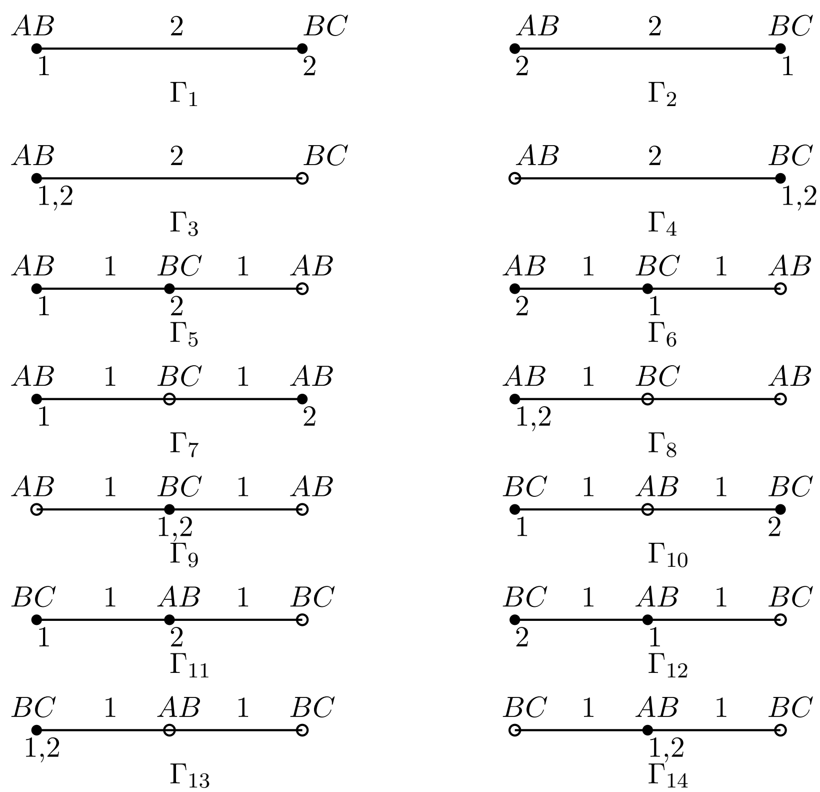
$< \beta_2, \beta_j >_{0,2,\beta} = 0$ *for* $j = 5, 7$ *and all* d*;* $< \beta_2, \beta_j >_{0,2,\beta} = 0$ *for* $j = 8, 9$ *and* $d \neq 1$*, and* $< \beta_2, \beta_8 >_{0,2,\beta_3} = < \beta_2, \beta_9 >_{0,2,\beta_3} = 1$*;*

$< \beta_3, \beta_j >_{0,2,\beta} = 0$ *for* $j = 4, 5, 7, 8, 9$ *and all* d*;* $< \beta_3, \beta_6 >_{0,2,\beta} = 0$ *for* $d \neq 1$*, and* $< \beta_3, \beta_6 >_{0,2,\beta_3} = 2$.

Proof. (i) If we take the invariant line between B_1 and B_2 as the representative for β_1 and the standard representatives of β_j, where we make free choices for f's and pt's, then we see that $< \beta_1, \beta_j >_{0,2,\beta} = 0$ for $j = 4, \cdots, 9$ and any d.

If we take the invariant line between AB and BC as the representative for β_2, then $< \beta_2, \beta_j >_{0,2,\beta} = 0$ for $j = 4, 5, 8, 9$ and any d, where in the standard representatives of β_5, β_8 and β_9, we take f to be f_∞ and pt to be the point D. Also $< \beta_2, \beta_6 >_{0,2,\beta} = < \beta_2, \beta_7 >_{0,2,\beta} = 0$, when $d \neq 2$.

So we need to consider the cases when $d = 2$. For $< \beta_2, \beta_6 >_{0,2,2\beta_2}$, there are nonzero terms in the localization formula from the fixed point loci described by the graphs:



Here we only work out the equivariant Euler classes of the normal bundles for Γ_1 and Γ_9. The Euler classes of the normal bundles of all other components can be computed either similarly or as before.

For Γ_1, $e^{\mathbb{C}^*}(B_1^m) = 1, e^{\mathbb{C}^*}(B_4^m) = 1$. Since the induced action on the invariant line from AB to

BC has weights $\frac{1}{2}\lambda, -\frac{1}{2}\lambda$ at the two end by Lemma 4.2.2, using equivariant K-theoretic localization, we have

$$\begin{aligned}\chi(f^*TF^{[2]}) &= \frac{t^{\lambda}+t^{\mu}+t^{\lambda}+t^{-\mu}}{1-t^{-\frac{1}{2}\lambda}} + \frac{t^{\lambda}+t^{-\mu}+t^{-\lambda}+t^{\lambda+\mu}}{1-t^{\frac{1}{2}\lambda}} \\ &= t^{\lambda}+t^{-\lambda}+t^{-\mu}+t^{\frac{1}{2}\lambda}+t^{-\frac{1}{2}\lambda}-t^{\frac{1}{2}\lambda+\mu}+1.\end{aligned}$$

From this we have

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu}.$$

For Γ_9, $e^{\mathbb{C}^*}(B_1^m) = \lambda^2, e^{\mathbb{C}^*}(B_4^m) = (-\lambda-e_3)(-\lambda-e_4) = (\lambda+e_3)(\lambda+e_4)$, where e_3, e_4 are the Euler classes of the respective cotangent line bundles over $\overline{\mathcal{M}}_{0,4}$, which correspond to the nodal points of the component represented by the vertex BC with the components represented by two edges from AB to BC. By equivariant K-theoretic localization and using normalization sequence as before we have

$$\chi(f^*TF^{[2]}) = 3t^{\lambda}+t^{-\lambda}+t^{-\mu}-t^{\lambda+\mu}+2.$$

So

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_9}) = \frac{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}{\lambda+\mu}.$$

All the equivariant Euler classes of the normal bundles are listed as follows:

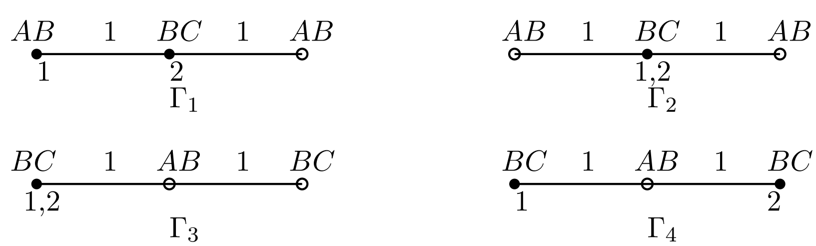
$$
\begin{aligned}
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu}, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = \frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu}, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4}) = \frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_5}) = \frac{\lambda^5\mu}{\lambda+\mu}, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_6}) = \frac{\lambda^5\mu}{\lambda+\mu},\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_7}) = -2\frac{\lambda^5\mu}{\lambda+\mu}, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_8}) = -2\frac{\lambda^5\mu}{\lambda+\mu},\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_9}) = \frac{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}{\lambda+\mu}, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_{10}}) = -2\lambda^5,\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_{11}}) = \lambda^5, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_{12}}) = \lambda^5,\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_{13}}) = -2\lambda^5, \qquad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_{14}}) = -\lambda^2(\lambda-e_3)(\lambda-e_4).
\end{aligned}
$$

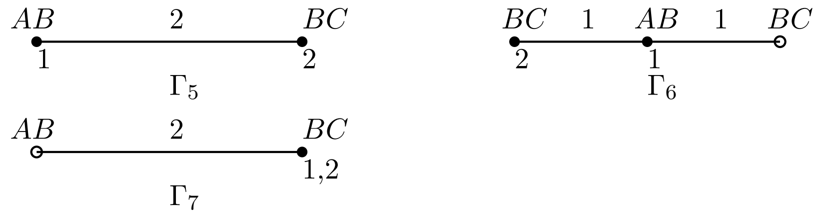
Using localization formula, we have

$$
\begin{aligned}
<\beta_2,\beta_6>_{0,1,2\beta_2} =& -2\frac{\lambda\mu^3(\lambda+\mu)(\lambda+2\mu)}{\lambda^5\mu} + \frac{\lambda\mu^4(\lambda+2\mu)}{\lambda^5\mu} + \frac{\lambda\mu^2(\lambda+\mu)^2(\lambda+2\mu)}{\lambda^5\mu}\\
&+ \frac{\lambda\mu^3(\lambda+\mu)^2}{\lambda^5\mu} + \frac{\lambda\mu^3(\lambda+\mu)^2}{\lambda^5\mu} + \frac{\lambda\mu^4(\lambda+\mu)}{-2\lambda^5\mu} + \frac{\lambda\mu^4(\lambda+\mu)}{-2\lambda^5\mu}\\
&+ \frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}} \frac{\lambda\mu^2(\lambda+\mu)^3}{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)} + \frac{\lambda\mu^2(\lambda+\mu)^2}{-2\lambda^5} + \frac{\lambda\mu^3(\lambda+\mu)}{\lambda^5}\\
&+ \frac{\lambda\mu^3(\lambda+\mu)}{\lambda^5} + \frac{\lambda\mu^2(\lambda+\mu)^2}{-2\lambda^5} - \frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}} \frac{\lambda\mu^4}{\lambda^2(\lambda-e_3)(\lambda-e_4)}\\
&= 0,
\end{aligned}
$$

where the factor 2 in front of the first term in the sum takes care of terms from Γ_1 and Γ_2. Here we used the fact that $\int_{\overline{\mathcal{M}}_{0,4}} e_3 = \int_{\overline{\mathcal{M}}_{0,4}} e_4 = 1$.

Let's turn to $<\beta_2,\beta_7>_{0,2,2\beta_2}$. We still take the invariant line between AB and BC for β_2. The nonzero terms in the localization formula from the fixed point loci are described by the graphs:





The equivariant Euler classes of their virtual normal bundles have all been worked out before, so we just list them below:

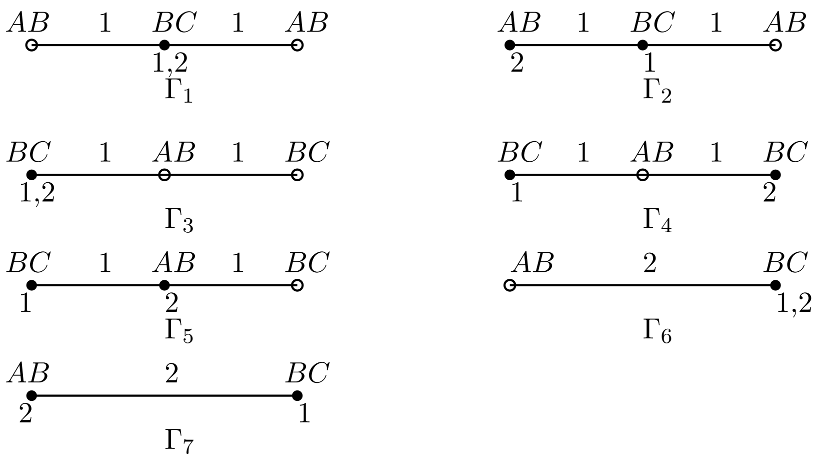
$$
\begin{aligned}
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = \frac{\lambda^5\mu}{\lambda+\mu}, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = \frac{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}{\lambda+\mu},\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4}) = -2\lambda^5, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_5}) = -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_6}) = \lambda^5, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_7}) = \frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},
\end{aligned}
$$

where e_3, e_4 are the Euler classes of the respective cotangent line bundles over $\overline{\mathcal{M}}_{0,4}$ explained above. So by localization formula,

$$
\begin{aligned}
<\beta_2,\beta_7>_{0,2,2\beta_2} =& \frac{-\lambda\mu^2(-\lambda^2)(\lambda+\mu)}{\lambda^5\mu} + \frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}} \frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)(\lambda+\mu)}{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}\\
&+\frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)}{-2\lambda^5} + \frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)}{-2\lambda^5} - \frac{-\lambda\mu^2(-\lambda^2)(\lambda+2\mu)}{\lambda^5\mu}\\
&+\frac{-\lambda\mu^2(-\lambda^2)}{\lambda^5} + \frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)(\lambda+2\mu)}{\lambda^5\mu}\\
=&0.
\end{aligned}
$$

If we take the invariant line between BC and BD for β_3 and the standard representatives for β_4, β_5 and β_8, where we take f to be f_∞, then $<\beta_3,\beta_j>_{0,2,\beta}=0$, for $j=4,5,8$ and any d; if we take the invariant line between AC and BC for β_3 and the standard representative for β_9 where we assign D to pt, then we see $<\beta_3,\beta_9>_{0,2,\beta}=0$. Now we keep the invariant line between BC and BD for β_3. It's not hard to see $<\beta_3,\beta_6>_{0,2,\beta}=<\beta_3,\beta_7>_{0,2,\beta}=0$, when $d\neq 2$.

When $d=2$, for $<\beta_3,\beta_6>_{0,2,\beta}$, the nonzero terms in the localization formula from the fixed point loci are described by the graphs:



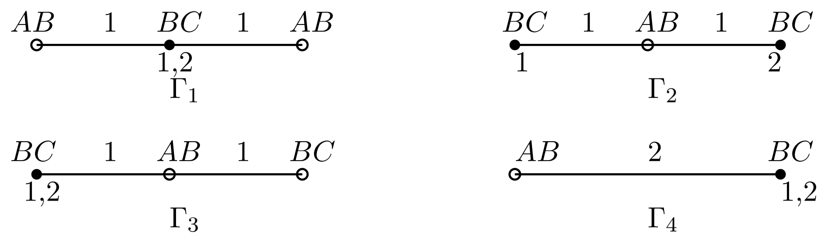
Again the equivariant Euler classes of their virtual normal bundles have all been worked out before.

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = \frac{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}{\lambda+\mu}, \quad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \frac{\lambda^5\mu}{\lambda+\mu},$$
$$e^{\mathbb{C}^*}(N_{\Gamma_3}^{vir}) = e^{\mathbb{C}^*}(N_{\Gamma_4}^{vir}) = -2\lambda^5, \quad e^{\mathbb{C}^*}(N_{\Gamma_5}^{vir}) = \lambda^5,$$
$$e^{\mathbb{C}^*}(N_{\Gamma_6}^{vir}) = \frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu}, \quad e^{\mathbb{C}^*}(N_{\Gamma_7}^{vir}) = -\frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},$$

where e_3, e_4 are the Euler classes of the respective cotangent line bundles over $\overline{\mathcal{M}}_{0,4}$. So by localization formula,

$$\begin{aligned}
<\beta_3,\beta_6>_{0,2,2\beta_2} =& \frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}} \frac{-\lambda^2\mu^2(\lambda+\mu)^2}{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)} + \frac{-\lambda^2\mu^3(\lambda+\mu)}{\lambda^5\mu} \\
&+ \frac{-\lambda^2\mu^2(\lambda+\mu)}{-2\lambda^5} + \frac{-\lambda^2\mu^2(\lambda+\mu)}{-2\lambda^5} + \frac{-\lambda^2\mu^3}{\lambda^5} \\
&+ \frac{-\lambda^2\mu^2(\lambda+\mu)(\lambda+2\mu)}{\lambda^5\mu} - \frac{-\lambda^2\mu^3(\lambda+2\mu)}{\lambda^5\mu} \\
=& 0.
\end{aligned}$$

For $<\beta_3,\beta_7>_{0,2,2\beta_2}$, we keep the representative for β_3. Then the connected components with nonzero terms in the localization formula are described by the graphs:



Their equivariant Euler classes are listed above.

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = \frac{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)}{\lambda+\mu}, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = -2\lambda^5,$$
$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = -2\lambda^5, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4}) = \frac{1}{2}\frac{\lambda^5\mu}{\lambda+2\mu},$$

where e_3, e_4 are the Euler classes of the respective cotangent line bundles over $\overline{\mathcal{M}}_{0,4}$. Substituting them in the localization formula, we get

$$< \beta_3, \beta_7 >_{0,2,2\beta_2} = \frac{1}{2}\int_{\overline{\mathcal{M}}_{0,4}} \frac{-\lambda^4\mu(\lambda+\mu)}{\lambda^2\mu(\lambda+e_3)(\lambda+e_4)} + \frac{-\lambda^4\mu}{-2\lambda^5}$$
$$+\frac{-\lambda^4\mu}{-2\lambda^5} + \frac{-\lambda^4\mu(\lambda+2\mu)}{\lambda^5\mu} = 0.$$

(ii) When $\beta = d\beta_1 + (\beta_3 - \beta_1)$, we take the invariant line between C_1 and C_2 for β_1 and the standard representatives for β_j, where we take f to be f_0 and pt to be A, then we see $< \beta_1, \beta_j >_{0,2,\beta} = 0$ for $j = 5, 7, 8, 9$ and any d.

Also, if we take the invariant line between AD and CD for β_2 and the standard representative for β_5, where f is taken to be f_0, then $< \beta_2, \beta_5 >_{0,2,\beta} = 0$ for any d.

Now we take the invariant line from AB to BC for the representative for β_2. Then $< \beta_2, \beta_7 >_{0,2,\beta} = 0$ if $d \neq 1$. When $d = 1$, the nonzero terms appearing in the localization formula are given by the connected components described by the following graphs:

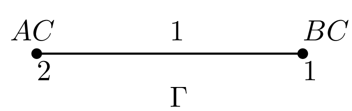
We have

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = \lambda^2\mu^2(\lambda+\mu), \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = -\lambda^2\mu^2(\lambda+\mu),$$
$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = -\lambda^2\mu(\lambda+\mu)^2, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4})) = \lambda^2\mu(\lambda+\mu)^2.$$

So

$$\begin{aligned} < \beta_2, \beta_7 >_{0,2,\beta} =& \frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)}{\lambda^2\mu^2(\lambda+\mu)} + \frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)}{-\lambda^2\mu^2(\lambda+\mu)} \\ &+ \frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)}{-\lambda^2\mu(\lambda+\mu)^2} + \frac{-\lambda\mu(\lambda+\mu)(-\lambda^2)}{\lambda^2\mu(\lambda+\mu)^2} \\ =&0. \end{aligned}$$

To compute $< \beta_2, \beta_8 >_{0,2,\beta}$, we still take the invariant line from AB to BC for the representative for β_2 and the standard representative for β_8, where f is taken to be f_∞. Then we see that $< \beta_2, \beta_8 >_{0,2,\beta} = 0$ if $d \neq 1$. When $d = 1$, there is only one nonzero term in the localization from the component described by the graph



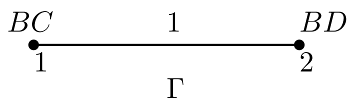
It's equivariant normal bundle is

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma}) = \lambda^2\mu^2(\lambda+\mu).$$

So

$$< \beta_2, \beta_8 >_{0,2,\beta} = \frac{-\lambda\mu(\lambda+\mu)(-\lambda\mu)}{\lambda^2\mu^2(\lambda+\mu)} = 1.$$

To compute $<\beta_2, \beta_9>_{0,2,\beta}$, we still take the invariant line from AB to BC for the representative for β_2 and the standard representative for β_9, where pt is taken to be D. Then we see that $<\beta_2, \beta_9>_{0,2,\beta}= 0$ if $d \neq 1$. When $d = 1$, there is only one nonzero term in the localization formula given by the connected component described the graph:

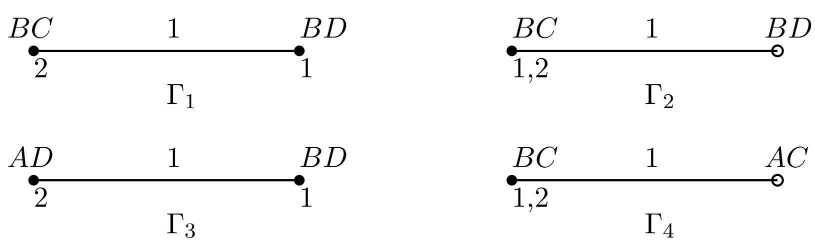


$e^{\mathbb{C}^*}(N_\Gamma^{vir}) = -\lambda^2\mu(\lambda+\mu)^2$, so

$$<\beta_2, \beta_9>_{0,2,\beta}=\frac{-\lambda\mu(\lambda+\mu)\lambda(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)^2} = 1.$$

If we take the invariant line between BC and BD for β_3, we see $<\beta_3, \beta_j>_{0,2,\beta}= 0$, for $j = 4, 5$ and all d, where we can take either f_0 or f_∞ for f in the representative of β_5.

Now we fix the representative for β_3 to be the invariant line between BC and BD. Then $<\beta_3, \beta_6>_{0,2,\beta}= 0$ for all $d \neq 1$. For $d = 1$, there are nonzero terms in the localization from the connected components described by the graphs:



We have

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = -\lambda^2\mu(\lambda+\mu)^2, \quad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \lambda^2\mu(\lambda+\mu)^2,$$
$$e^{\mathbb{C}^*}(N_{\Gamma_3}^{vir}) = -\lambda^2\mu^2(\lambda+\mu), \quad e^{\mathbb{C}^*}(N_{\Gamma_4}^{vir}) = -\lambda^2\mu^2(\lambda+\mu),$$

So

$$
\begin{aligned}
<\beta_3, \beta_6>_{0,2,\beta} =& \frac{-\lambda^2\mu^2(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)^2} + \frac{-\lambda^2\mu^2(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)^2} \\
&+ \frac{-\lambda^2\mu^2(\lambda+\mu)}{-\lambda^2\mu^2(\lambda+\mu)} + \frac{-\lambda^2\mu\mu(\lambda+\mu)}{-\lambda^2\mu^2(\lambda+\mu)} \\
=& 2.
\end{aligned}
$$

To compute $<\beta_3, \beta_j>_{0,2,\beta}$ for $j = 7, 8, 9$, we take the invariant line from CD to D_2 for $\beta_3 - \beta_1$ and the standard representatives for β_j, where f is taken to be f_0 and pt to be B. Then we see that

$$<\beta_3 - \beta_1, \beta_j>_{0,2,\beta} = 0$$

for any d. But $<\beta_1, \beta_j>_{0,2,\beta} = 0$, so $<\beta_3, \beta_j>_{0,2,\beta} = 0$ for $j = 7, 8, 9$. □

In this proposition, four sequences of invariants are not treated, which are $<\beta_1, \beta_4>_{0,2,\beta}$, $<\beta_2, \beta_4>_{0,2,\beta}$, $<\beta_1, \beta_6>_{0,2,\beta}$ and $<\beta_2, \beta_6>_{0,2,\beta}$ for $\beta = d\beta_1 + (\beta_3 - \beta_1)$, because they involve higher degrees on β_1. They will be determined in the next chapter.

For the last pair $(d_2, d_3) = (1, 0)$, the virtual dimension of the moduli space is equal to 4. The degree decomposition of the two insertions has to be $1 + 3$, or $2 + 2$. For the first type, we have

Proposition 4.4.8. *For $\beta = d\beta_1 + (\beta_2 - \beta_1)$,*

(i) $<\beta_1, \beta_j>_{0,2,\beta} = 0$, for $j = 10, 11, 12$ and all d;

(ii) $<\beta_i, \beta_j>_{0,2,\beta} = 0$, for $i = 2, 3$, $j = 10, 11, 12$ and $d \neq 1$; when $d = 1$,

$$<\beta_2, \beta_{10}>_{0,2,\beta_2} = 0, \quad <\beta_2, \beta_{11}>_{0,2,\beta_2} = 1, \quad <\beta_2, \beta_{12}>_{0,2,\beta_2} = -1;$$

$$<\beta_3, \beta_{10}>_{0,2,\beta_2} = 0, \quad <\beta_3, \beta_{11}>_{0,2,\beta_2} = -1, \quad <\beta_3, \beta_{12}>_{0,2,\beta_2} = 1.$$

Proof. By Axiom of Divisors and recalling Proposition 3.3.2, we only need to consider the cases when $d = 1$. First,

$$<\beta_2, \beta_{10}>_{0,2,\beta_2} = \int_{\beta_2} \beta_{10} <\beta_2>_{0,1,\beta_2} = 0,$$

because the intersection product of β_2 and β_{10} is 0. Similarly,

$$< \beta_2, \beta_{11} >_{0,2,\beta_2} = \int_{\beta_2} \beta_{11} < \beta_2 >_{0,1,\beta_2} = (-1)(-1) = 1,$$
$$< \beta_2, \beta_{12} >_{0,2,\beta_2} = \int_{\beta_2} \beta_{12} < \beta_2 >_{0,1,\beta_2} = 1(-1) = -1,$$
$$< \beta_3, \beta_{10} >_{0,2,\beta_2} = \int_{\beta_2} \beta_{10} < \beta_3 >_{0,1,\beta_2} = 0,$$

because the intersection product of β_2 and β_{10} is 0. Similarly,

$$< \beta_3, \beta_{11} >_{0,2,\beta_2} = \int_{\beta_2} \beta_{11} < \beta_3 >_{0,1,\beta_2} = (-1) \cdot 1 = -1,$$
$$< \beta_3, \beta_{12} >_{0,2,\beta_2} = \int_{\beta_2} \beta_{12} < \beta_3 >_{0,1,\beta_2} = 1 \cdot 1 = 1.$$

Here we make use the intersection product table in §3.2. □

When $(d_2, d_3) = (1, 0)$, we have the second type of degree decomposition of the two insertions $2 + 2$.

Proposition 4.4.9. *For $\beta = d\beta_1 + (\beta_2 - \beta_1)$, we have*

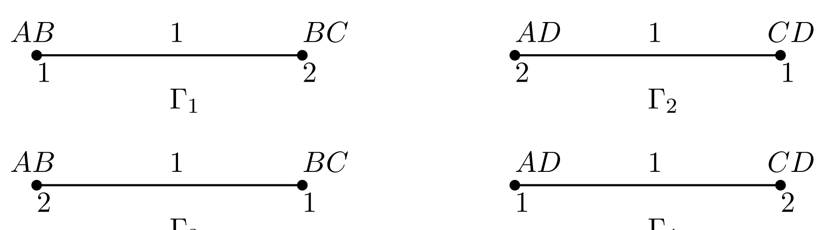
(i) $< \beta_4, \beta_6 >_{0,2,\beta} = < \beta_5, \beta_6 >_{0,2,\beta} = 0$ for any d;

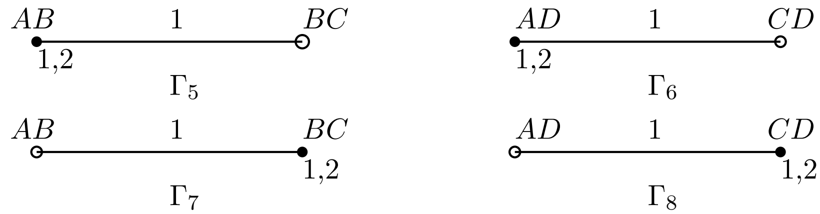
(ii) $< \beta_6, \beta_k >_{0,2,\beta} = 0$ for $k = 6, 7, 8$ and $d \neq 1$, but $< \beta_6, \beta_6 >_{0,2,\beta_2} = 1, < \beta_6, \beta_7 >_{0,2,\beta_2} = -2, < \beta_6, \beta_8 >_{0,2,\beta_2} = 1$;

(iii) $< \beta_k, \beta_9 >_{0,2,\beta} = 0$ for $k = 4, \cdots, 9$ and any d.

Proof. (i) If we take the standard representatives for β_4, β_5 and β_6, where f is taken to be f_0 for β_5, then we see that $< \beta_4, \beta_6 >_{0,2,\beta} = < \beta_5, \beta_6 >_{0,2,\beta} = 0$.

(ii) Also we can see when $d \neq 1$, $< \beta_6, \beta_6 >_{0,2,\beta} = 0$. Now we compute $< \beta_6, \beta_6 >_{0,2,\beta}$ when $d = 1$. There are eight nonzero terms in the localization formula from fixed point loci described by the graphs:





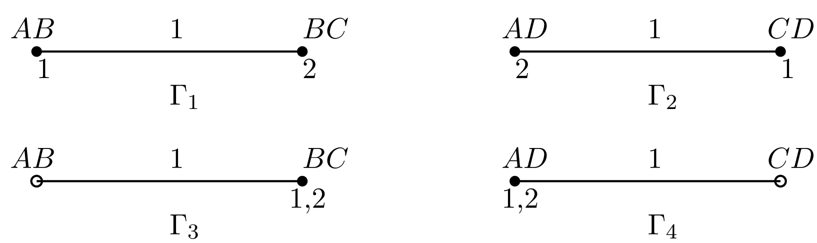
Their equivariant Euler classes are

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = \lambda^3\mu, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4}) = -\lambda^3(\lambda+\mu),$$
$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_5}) = e^{\mathbb{C}^*}(N^{vir}_{\Gamma_7}) = -\lambda^3\mu, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_6}) = e^{\mathbb{C}^*}(N^{vir}_{\Gamma_8}) = \lambda^3(\lambda+\mu).$$

Applying localization formula, we get

$$\begin{aligned}< \beta_6, \beta_6 >_{0,2,\beta_2} =& 2\frac{-\mu^2(-\mu)(\lambda+\mu)}{\lambda^3\mu} + 2\frac{\mu(\lambda+\mu)(\lambda+\mu)^2}{-\lambda^3(\lambda+\mu)} + \frac{(-\mu^2)^2}{-\lambda^3\mu} \\ &+ \frac{(-\mu(\lambda+\mu))^2}{\lambda^3(\lambda+\mu)} + \frac{(-\mu(\lambda+\mu))^2}{-\lambda^3\mu} + \frac{(-(\lambda+\mu)^2)^2}{\lambda^3(\lambda+\mu)} \\ =& 1.\end{aligned}$$

Using the standard representative for β_7, we see when $d \neq 1$, $< \beta_6, \beta_7 >_{0,2,\beta} = 0$. When $d = 1$, nonzero terms in the localization formula are given by the fixed point locus described by the graphs:



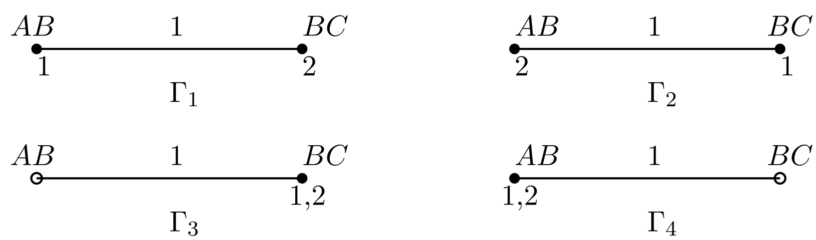
We have

$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = \lambda^3\mu, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = -\lambda^3(\lambda+\mu),$$
$$e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = -\lambda^3\mu, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4}) = \lambda^3(\lambda+\mu).$$

So

$$
\begin{aligned}
< \beta_6, \beta_7 >_{0,2,\beta_2} = & \frac{-\mu^2(-\lambda^2)}{\lambda^3\mu} + \frac{-(\lambda+\mu)^2(-\lambda^2)}{-\lambda^3(\lambda+\mu)} \\
& + \frac{\lambda^2\mu(\lambda+\mu)}{-\lambda^3\mu} + \frac{\lambda^2\mu(\lambda+\mu)}{\lambda^3(\lambda+\mu)} \\
= & -2.
\end{aligned}
$$

We take f_0 for f in the representative for β_8. Then when $d \neq 1, < \beta_6, \beta_8 >_{0,2,\beta} = 0$. When $d = 1$, there are nonzero terms in the localization formula from fixed point loci given by the graphs:



We have

$$
e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1}) = e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2}) = \lambda^3\mu, \quad e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3}) = e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4}) = -\lambda^3\mu.
$$

So

$$
\begin{aligned}
< \beta_6, \beta_8 >_{0,2,\beta} = & \frac{-\mu^2\lambda(\lambda+\mu)}{\lambda^3\mu} + \frac{-\mu(\lambda+\mu)\lambda\mu}{\lambda^3\mu} \\
& + \frac{-\mu(\lambda+\mu)\lambda(\lambda+\mu)}{-\lambda^3\mu} + \frac{-\mu^2\lambda\mu}{-\lambda^3\mu} \\
= & 1.
\end{aligned}
$$

(iii) If we use the above representative for β_8 and the standard representative for β_9, where pt is taken to be D, then we see $< \beta_8, \beta_9 >_{0,2,\beta} = 0$ for any d. If we take a different representative for β_9, where pt is taken to be B, then we see $< \beta_9, \beta_9 >_{0,2,\beta} = 0$ for any d. Also, $< \beta_4, \beta_9 >_{0,2,\beta} = < \beta_5, \beta_9 >_{0,2,\beta} = 0$ for any d, where we use f_∞ for f and B for pt in the standard representatives for β_5, β_9.

We keep the representative for β_9, where pt is taken to B. Then we see when $d \neq 1$, $< \beta_6, \beta_9 >_{0,2,\beta} = < \beta_7, \beta_9 >_{0,2,\beta} = 0$. When $d = 1$, for $< \beta_6, \beta_9 >_{0,2,\beta}$, the connected components appearing as nonzero terms in the localization formula are the same as for $< \beta_6, \beta_8 >_{0,2,\beta}$. Everything has been worked out.

$$
\begin{aligned}
< \beta_6, \beta_9 >_{0,2,\beta} =& \frac{-\mu^2(-\lambda\mu)}{\lambda^3\mu} + \frac{-\mu(\lambda+\mu)(-\lambda\mu)}{\lambda^3\mu} \\
&+ \frac{-\mu(\lambda+\mu)(-\lambda\mu)}{-\lambda^3\mu} + \frac{-\mu^2(-\lambda\mu)}{-\lambda^3\mu} \\
=& 0.
\end{aligned}
$$

For $< \beta_7, \beta_9 >_{0,2,\beta}$, we have connected components

The virtual normal bundles have been decided before, i.e.

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = -\lambda^3\mu, \quad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \lambda^3\mu,$$

so

$$< \beta_7, \beta_9 >_{0,2,\beta} = \frac{-\lambda^2(-\lambda\mu)}{-\lambda^3\mu} + \frac{-\lambda^2(-\lambda\mu)}{\lambda^3\mu} = 0.$$

$\square$

In the proof of this proposition, if we use the cycle $P(TF^{[2]}|_{S_0})$ to define a new class in $A_2(F^{[2]})$, denoted as γ, then we see $< \beta_5, \gamma >_{0,2,\beta} = < \gamma, \gamma >_{0,2,\beta} = 0$. But it's east to see $\gamma = \beta_4 + \beta_5$, so we get $< \beta_4, \beta_4 >_{0,2,\beta} = - < \beta_4, \beta_5 >_{0,2,\beta} = < \beta_5, \beta_5 >_{0,2,\beta}$. Also we have relations $< \gamma, \beta_7 >_{0,2,\beta} = < \gamma, \beta_8 >_{0,2,\beta} = 0$, so $< \beta_4, \beta_7 >_{0,2,\beta} = - < \beta_5, \beta_7 >_{0,2,\beta}$ and $< \beta_4, \beta_8 >_{0,2,\beta} = - < \beta_5, \beta_8 >_{0,2,\beta}$. We'll come back to these invariants together with $< \beta_4, \beta_4 >_{0,2,\beta}, < \beta_4, \beta_5 >_{0,2,\beta}, < \beta_5, \beta_5 >_{0,2,\beta}$ and $< \beta_7, \beta_7 >_{0,2,\beta}, < \beta_7, \beta_8 >_{0,2,\beta}, < \beta_8, \beta_8 >_{0,2,\beta}$ in the next chapter.

We could extend this method to compute three-pointed GW-invariants, but because of the existence of higher dimensional families of invariant lines, we only succeed in calculating a small number of them. In stead of carrying on this method for partial results, we are especially interested in the three-pointed GW-invariants with β_9 as two insertions for their usefulness in the computation of quantum products.

4.5 Computations of Some Three-Point Gromov-Witten Invariants

Since the virtual dimension of $\overline{\mathcal{M}}_{0,3}(F^{[2]}, \beta)$ is $d_2 + 2d_3 + 4$ for any curve class $\beta = d_1\beta_1 + d_2(\beta_2 - \beta_1) + d_3(\beta_3 - \beta_1)$, when two insertions are β_9, we must have the third insertion to be of degree $4 - (d_2 + 2d_3)$ as a Chow class. When $(d_2, d_3) = (4, 0), (2, 1), (0, 2)$, this insertion must be a point class; when $(d_2, d_3) = (3, 0), (1, 1)$, it is of degree 1; when $(d_2, d_3) = (2, 0), (0, 1)$, it is of degree 2; finally, when $(d_2, d_3) = (1, 0)$, it is of degree 3. We study them case by case in this section.

First the case when $(d_2, d_3) = (4, 0), (2, 1), (0, 2)$ is treated in the following

Proposition 4.5.1. *For $\beta = d\beta_1 + 4(\beta_2 - \beta_1), d\beta_1 + 2(\beta_2 - \beta_1) + (\beta_3 - \beta_1)$, and $d\beta_1 + 2(\beta_3 - \beta_1)$ for any d,*

$$< pt, \beta_9, \beta_9 >_\beta = 0.$$

Proof. For β_9 in the second and third insertions, we take the standard representative, where pt is taken to be B and D respectively.

For $\beta = d\beta_1 + 4(\beta_2 - \beta_1)$, if we take the point BD for the point class, then $< pt, \beta_9, \beta_9 >_\beta = 0$.

If we take the point D_1 for the point class, then we see that $< pt, \beta_9, \beta_9 >_\beta = 0$, for $\beta = d\beta_1 + 2(\beta_2 - \beta_1) + (\beta_3 - \beta_1), d\beta_1 + 2(\beta_3 - \beta_1)$. □

For the pair $(d_2, d_3) = (3, 0)$, we have

Proposition 4.5.2. *For $\beta = d\beta_1 + 3(\beta_2 - \beta_1)$,*

$$< \beta_i, \beta_9, \beta_9 >_\beta = 0,$$

for $i = 1, 2, 3$ and any d.

Proof. We keep the two representatives for β_9 and choose the line from A_1 to A_2 for β_1, the line from AB to BC for β_2, the line from AC to BC for β_3. Then the result follows. $\square$

For the pair $(d_2, d_3) = (1, 1)$, we have

Proposition 4.5.3. *For* $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$,

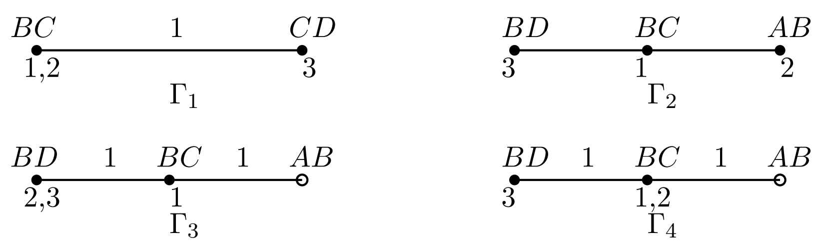
$< \beta_1, \beta_9, \beta_9 >_\beta = < \beta_2, \beta_9, \beta_9 >_\beta = 0$ *for any* d,

$< \beta_3, \beta_9, \beta_9 >_\beta = 0$, *when* $d \neq 2$; $< \beta_3, \beta_9, \beta_9 >_{\beta_2+\beta_3} = 1$.

Proof. If we choose the invariant line from A_1 to A_2 for β_1, then $< \beta_1, \beta_9, \beta_9 >_\beta = 0$.

If we choose the invariant line from A_1 to AC for the representative of $\beta_2 - \beta_1$, then we see $< \beta_2 - \beta_1, \beta_9, \beta_9 >_\beta = 0$, so $< \beta_2, \beta_9, \beta_9 >_\beta = 0$ from above.

To compute $< \beta_3, \beta_9, \beta_9 >_\beta$, we take the line from BC to AC for β_3. Then $< \beta_3, \beta_9, \beta_9 >_\beta =$ 0 when $d \neq 2$. When $d = 2$, there are nonzero terms in the localization from the connected components represented by the graphs:



Their equivariant Euler classes are

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = \lambda^4\mu(\lambda+\mu)^2, \qquad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \lambda^4\mu(\lambda+\mu)^2,$$
$$e^{\mathbb{C}^*}(N_{\Gamma_3}^{vir}) = -\lambda^3\mu(\lambda+\mu)^3, \quad e^{\mathbb{C}^*}(N_{\Gamma_4}^{vir}) = \lambda^2\mu(\lambda+\mu)(\lambda+\mu-e_3)(\lambda+e_4).$$

where e_3, e_4 are the Euler classes of the respective cotangent line bundles over $\overline{\mathcal{M}}_{0,4}$ corresponding to the nodal points of the component represented by the vertex BC with the components represented by edges from BC to BD and from BC to AB. So

$$\begin{aligned}
<\beta_3,\beta_9,\beta_9>_{\beta_2+\beta_3} &= \frac{-\lambda^2(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^4\mu(\lambda+\mu)^2} \\
&+ \frac{-\lambda^2(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^4\mu(\lambda+\mu)^2} + \frac{-\lambda^2(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{-\lambda^3\mu(\lambda+\mu)^3} \\
&+ \int_{\overline{\mathcal{M}}_{0,4}} \frac{-\lambda^2(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)(\lambda+\mu-e_3)(\lambda+e_4)} \\
&= 1.
\end{aligned}$$

Here we used the fact that $\int_{\overline{\mathcal{M}}_{0,4}} e_3 = \int_{\overline{\mathcal{M}}_{0,4}} e_4 = 1$. $\square$

The case when $(d_2, d_3) = (2, 0)$ is studied in the following

Proposition 4.5.4. *For $\beta = d\beta_1 + 2(\beta_2 - \beta_1)$,*

$$<\beta_k,\beta_9,\beta_9>_\beta = 0,$$

for $k = 4, \cdots, 9$ and any d.

Proof. We still use the two representatives for β_9 as above and take the standard representatives for the classes $\beta_4, \cdots, \beta_9$, where we take f to be f_0 and pt to be B. Then there is no nodal curve connecting the three cycles of the class β. So the GW-invariants are trivial. $\square$

The case when $(d_2, d_3) = (0, 1)$ is studied in the following

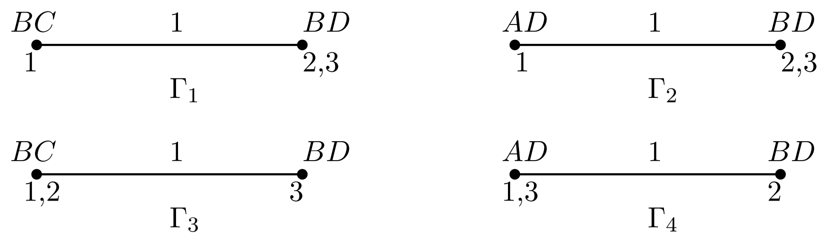
Proposition 4.5.5. *For $\beta = d\beta_1 + (\beta_3 - \beta_1)$,*

$$<\beta_k,\beta_9,\beta_9>_\beta = 0,$$

for $k = 4, \cdots, 9$ and any d.

Proof. We keep the standard representatives for the classes $\beta_4, \cdots, \beta_9$ as in the proof of the previous proposition. Then it is easy to see that $<\beta_4,\beta_9,\beta_9>_\beta = <\beta_5,\beta_9,\beta_9>_\beta = 0$ for any d.

When $d \neq 1$, $<\beta_6,\beta_9,\beta_9>_\beta = 0$. For $<\beta_6,\beta_9,\beta_9>_{\beta_3}$, there are four connected components represented by the graphs:



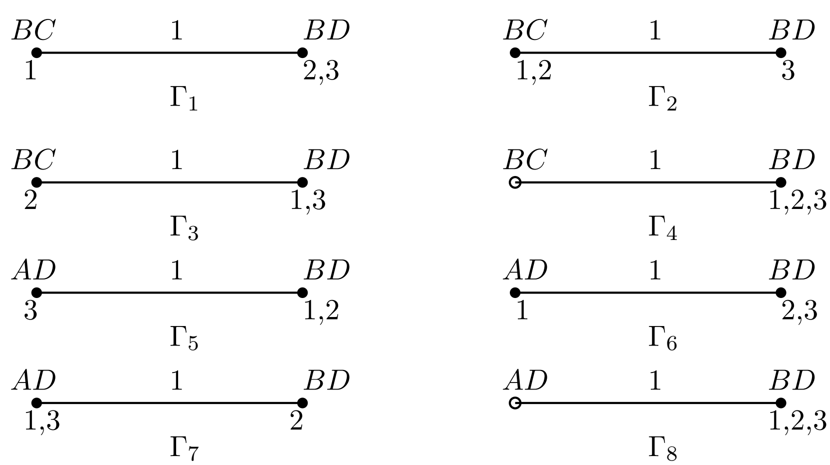
Their equivariant Euler classes are

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = \lambda^2\mu(\lambda+\mu)^3, \qquad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = \lambda^2\mu^3(\lambda+\mu),$$
$$e^{\mathbb{C}^*}(N_{\Gamma_3}^{vir}) = -\lambda^2\mu(\lambda+\mu)^3, \quad e^{\mathbb{C}^*}(N_{\Gamma_4}^{vir}) = -\lambda^2\mu^3(\lambda+\mu).$$

So

$$\begin{aligned} < \beta_6, \beta_9, \beta_9 >_{\beta_3} =& \frac{-\mu(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)^3} + \frac{-\mu(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu^3(\lambda+\mu)} \\ &+ \frac{-\mu(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)^3} + \frac{-\mu(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{-\lambda^2\mu^3(\lambda+\mu)} \\ =& 0. \end{aligned}$$

Again when $d \neq 1$, $< \beta_7, \beta_9, \beta_9 >_\beta = 0$. When $d = 1$, there are eight connected components represented by the graphs in the localization:



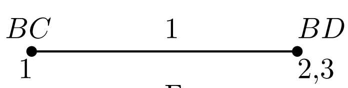
Their equivariant Euler classes are

$$\begin{aligned}
e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) &= \lambda^2\mu(\lambda+\mu)^3, \qquad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = -\lambda^2\mu(\lambda+\mu)^3,\\
e^{\mathbb{C}^*}(N_{\Gamma_3}^{vir}) &= \lambda^2\mu(\lambda+\mu)^3, \qquad e^{\mathbb{C}^*}(N_{\Gamma_4}^{vir}) = \lambda^2\mu(\lambda+\mu)(\lambda+\mu+e_4),\\
e^{\mathbb{C}^*}(N_{\Gamma_5}^{vir}) &= \lambda^2\mu^3(\lambda+\mu), \qquad e^{\mathbb{C}^*}(N_{\Gamma_6}^{vir}) = \lambda^2\mu^3(\lambda+\mu),\\
e^{\mathbb{C}^*}(N_{\Gamma_7}^{vir}) &= -\lambda^2\mu^3(\lambda+\mu), \qquad e^{\mathbb{C}^*}(N_{\Gamma_8}^{vir}) = \lambda^2\mu(\lambda+\mu)(\mu+e_4),
\end{aligned}$$

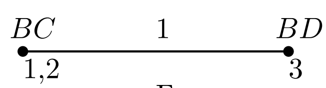
where e_4 in $e^{\mathbb{C}^*}(N_{\Gamma_4}^{vir})$ is the Euler class of the cotangent line bundle over $\overline{\mathcal{M}}_{0,4}$ corresponding to the nodal point of the component represented by the vertex BD with the component represented by the edge from BD to BC and where e_4 in $e^{\mathbb{C}^*}(N_{\Gamma_8}^{vir})$ is that corresponding to the nodal point of the component represented by the vertex BD with the component represented by the edge from BD to AD. So

$$\begin{aligned}
<\beta_7,\beta_9,\beta_9>_{\beta_3}=&\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)^3}+\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)^3}\\
&+\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)^3}+\int_{\overline{\mathcal{M}}_{0,4}}\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)(\lambda+\mu+e_4)}\\
&+\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu^3(\lambda+\mu)}+\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu^3(\lambda+\mu)}\\
&+\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{-\lambda^2\mu^3(\lambda+\mu)}+\int_{\overline{\mathcal{M}}_{0,4}}\frac{-\lambda^2(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)(\mu+e_4)}\\
=&0.
\end{aligned}$$

When $d \neq 1$, $<\beta_8,\beta_9,\beta_9>_\beta= 0$. When $d = 1$, there are two connected components represented by the graphs in the localization:



Γ_1



Γ_2

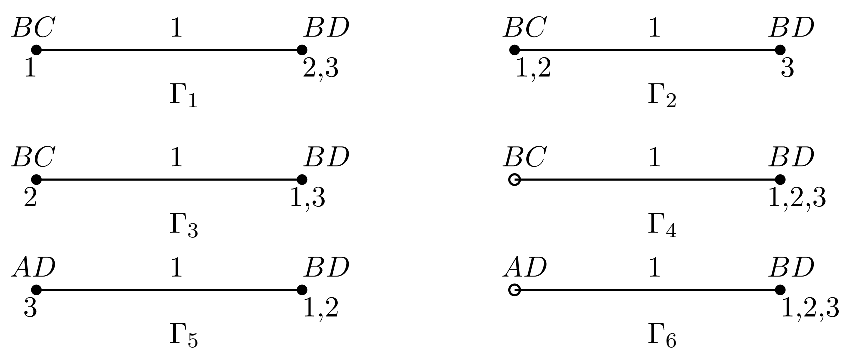
Their equivariant Euler classes are

$$e^{\mathbb{C}^*}(N_{\Gamma_1}^{vir}) = \lambda^2\mu(\lambda+\mu)^3, \qquad e^{\mathbb{C}^*}(N_{\Gamma_2}^{vir}) = -\lambda^2\mu(\lambda+\mu)^3.$$

So

$$
\begin{aligned}
<\beta_8,\beta_9,\beta_9>_{\beta_3}=&\frac{\lambda(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)^3}+\frac{\lambda(\lambda+\mu)(-\lambda\mu)\lambda(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)^3}\\
=&0.
\end{aligned}
$$

We recall that we take pt to be B for the representatives of the first two β_9 and D for the representative of the third β_9. When $d\neq 1$, $<\beta_9,\beta_9,\beta_9>_\beta=0$. When $d=1$, there are nonzero terms in the localization from the connected components represented by the graphs:



Their equivariant Euler classes are

$$
\begin{aligned}
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_1})=\lambda^2\mu(\lambda+\mu)^3, &&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_2})=-\lambda^2\mu(\lambda+\mu)^3,\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_3})=\lambda^2\mu(\lambda+\mu)^3, &&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4})=\lambda^2\mu(\lambda+\mu)(\lambda+\mu+e_4),\\
&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_5})=\lambda^2\mu^3(\lambda+\mu), &&e^{\mathbb{C}^*}(N^{vir}_{\Gamma_6})=\lambda^2\mu(\lambda+\mu)(\mu+e_4),
\end{aligned}
$$

where e_4 in $e^{\mathbb{C}^*}(N^{vir}_{\Gamma_4})$ is the Euler class of the cotangent line bundle over $\overline{\mathcal{M}}_{0,4}$ corresponding to the nodal point of the component represented by the vertex BD with the component represented by the edge from BD to BC and where e_4 in $e^{\mathbb{C}^*}(N^{vir}_{\Gamma_6})$ is that corresponding to the nodal point of the component represented by the vertex BD with the component represented by the edge from

BD to AD. So

$$\begin{aligned}<\beta_9,\beta_9,\beta_9>_{\beta_3}=&\frac{(-\lambda\mu)^2\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)^3}+\frac{(-\lambda\mu)^2\lambda(\lambda+\mu)}{-\lambda^2\mu(\lambda+\mu)^3}\\&+\frac{(-\lambda\mu)^2\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)^3}+\int_{\overline{\mathcal{M}}_{0,4}}\frac{(-\lambda\mu)^2\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)(\lambda+\mu+e_4)}\\&+\frac{(-\lambda\mu)^2\lambda(\lambda+\mu)}{\lambda^2\mu^3(\lambda+\mu)}+\int_{\overline{\mathcal{M}}_{0,4}}\frac{(-\lambda\mu)^2\lambda(\lambda+\mu)}{\lambda^2\mu(\lambda+\mu)(\mu+e_4)}\\=&0.\end{aligned}$$

□

Finally, when $(d_2,d_3)=(1,0)$, we have the following

Proposition 4.5.6. *For* $\beta=d\beta_1+(\beta_2-\beta_1)$,

$$<\beta_j,\beta_9,\beta_9>_\beta=0,$$

for $j=10,11,12$ *and any* d.

Proof. By Axiom of Divisors,

$$<\beta_j,\beta_9,\beta_9>_\beta=\int_\beta\beta_j<\beta_9,\beta_9>_\beta.$$

These invariants all vanish since $<\beta_9,\beta_9>_\beta$ vanishes by Proposition 4.4.9. □

Chapter 5

Quantum Cohomology Ring

In this chapter, we come to the last step toward our goal, which is the determination of quantum cohomology ring structure of the Hilbert scheme. For this purpose, sufficiently many products of basis elements have to be calculated. From the results in the previous chapter, quantum products of generators can be decided, which is the topic of the first section.

5.1 Some Quantum Products

We have determined a presentation of the Chow ring $H^*(F^{[2]})$ generated by $\beta_9, \beta_{10}, \beta_{11}, \beta_{12}$ with the relations:

$$
\begin{aligned}
&P1: \ \beta_{10}^2 - 2\beta_{10}\beta_{11} - 3\beta_{10}\beta_{12} + 2\beta_{12}^2 + 4\beta_{11}\beta_{12} = 0,\\
&P2: \ \beta_{10}\beta_{12}^2 = 0,\\
&P3: \ \beta_{12}^3 = 0,\\
&P4: \ \beta_{11}\beta_{12}^2 - 2\beta_9\beta_{12} = 0,\\
&P5: \ \beta_{10}\beta_{11}^2 + 2\beta_9\beta_{10} = 0,\\
&P6: \ \beta_{10}\beta_{11}\beta_{12} - 2\beta_9\beta_{10} = 0,\\
&P7: \ \beta_{11}^2\beta_{12} - 2\beta_9\beta_{11} + \beta_9\beta_{12} = 0,\\
&P8: \ \beta_{11}^3 + 3\beta_9\beta_{11} = 0,\\
&P9: \ \beta_9\beta_{11}\beta_{12} - \beta_9^2 = 0.
\end{aligned}
$$

The dual basis of our standard basis $\beta_0, \beta_1, \cdots, \beta_{12}, \beta_{13}$ is

$$\beta_{13},\ -\frac{1}{2}\beta_{10},\ \beta_{12},\ \beta_{11}+\beta_{12},\ -\frac{1}{2}\beta_5,\ -\frac{1}{2}\beta_4-\frac{1}{2}\beta_5,\ \frac{1}{2}\beta_7,$$
$$\frac{1}{2}\beta_6+\frac{1}{2}\beta_7+\frac{1}{2}\beta_8,\ \frac{1}{2}\beta_7+\beta_8,\ \beta_9,\ -\frac{1}{2}\beta_1,\ \beta_3,\ \beta_2+\beta_3,\ \beta_0.$$

With the computational results in §4.3 and §4.4, quantum products from $\beta_{10}, \beta_{11}, \beta_{12}$ can be computed. First, by definition,

$$\begin{aligned}
\beta_{10} * \beta_{10} =& \beta_{10}^2 + \sum_{\beta\neq 0}\sum_i < \beta_{10}, \beta_{10}, T_i >_\beta q^\beta T^i \\
=& \beta_{10}^2 + \sum_d < \beta_{10}, \beta_{10}, pt >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 \\
&+ \sum_d < \beta_{10}, \beta_{10}, pt >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 \\
&+ \sum_d < \beta_{10}, \beta_{10}, \beta_1 >_{d\beta_1+(\beta_2-\beta_1)} q_1^d q_2 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_{10}, \beta_{10}, \beta_2 >_{d\beta_1+(\beta_2-\beta_1)} q_1^d q_2 \beta_{12} \\
&+ \sum_d < \beta_{10}, \beta_{10}, \beta_3 >_{d\beta_1+(\beta_2-\beta_1)} q_1^d q_2 (\beta_{11}+\beta_{12}) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_4 >_{d\beta_1} q_1^d (-\frac{1}{2}\beta_5) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_5 >_{d\beta_1} q_1^d (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_6 >_{d\beta_1} q_1^d (\frac{1}{2}\beta_7) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_7 >_{d\beta_1} q_1^d (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_8 >_{d\beta_1} q_1^d (\frac{1}{2}\beta_7 + \beta_8) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_9 >_{d\beta_1} q_1^d \beta_9
\end{aligned}$$

$$
\begin{aligned}
=&\beta_{10}^2 + \sum_d < \beta_{10}, \beta_{10}, pt >_{d\beta_1+(\beta_3-\beta_1)} \, q_1^d q_3 \\
&+ \sum_d < \beta_{10}, \beta_{10}, \beta_2 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 \beta_{12} \\
&+ \sum_d < \beta_{10}, \beta_{10}, \beta_3 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (\beta_{11} + \beta_{12}) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_4 >_{d\beta_1} \, q_1^d (-\frac{1}{2}\beta_5) \\
&+ \sum_{d\neq 0} < \beta_{10}, \beta_{10}, \beta_5 >_{d\beta_1} \, q_1^d (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
=&\beta_{10}^2 + \int_{\beta_3} \beta_{10} \int_{\beta_3} \beta_{10} \; < pt >_{\beta_3} q_1 q_3 \\
&+ \int_{\beta_2} \beta_{10} \int_{\beta_2} \beta_{10} \; < \beta_2 >_{\beta_2} q_1 q_2 \beta_{12} \\
&+ \int_{\beta_2} \beta_{10} \int_{\beta_2} \beta_{10} \; < \beta_3 >_{\beta_2} q_1 q_2 (\beta_{11} + \beta_{12}) \\
&+ \sum_{d\neq 0} \int_{d\beta_1} \beta_{10} \int_{d\beta_1} \beta_{10} \; < \beta_4 >_{d\beta_1} q_1^d (-\frac{1}{2}\beta_5) \\
&+ \sum_{d\neq 0} \int_{d\beta_1} \beta_{10} \int_{d\beta_1} \beta_{10} \; < \beta_5 >_{d\beta_1} q_1^d (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
=&\beta_{10}^2 + \sum_{d\neq 0} (-2d)^2 \frac{-2}{d} q_1^d (-\frac{1}{2}\beta_5) + \sum_{d\neq 0} (-2d)^2 \frac{-4}{d} q_1^d (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
=&\beta_{10}^2 + 8 \sum_{d\neq 0} d q_1^d \beta_4 + 12 \sum_{d\neq 0} d q_1^d \beta_5.
\end{aligned}
$$

Here above, many terms vanish because either the invariants involved are trivial or the integrals $\int_{\beta_2} \beta_{10} = \int_{\beta_3} \beta_{10} = 0$. We also used the fact that $\int_{\beta_1} \beta_{10} = -2$.

Similarly, if we substitute β_{11} for the second β_{10} in the expression of $\beta_{10} * \beta_{10}$ and omit trivial terms for the same reasons stated above, we have

$$
\begin{aligned}
\beta_{10} * \beta_{11} =&\beta_{10}\beta_{11} + \sum_d < \beta_{10}, \beta_{11}, pt >_{d\beta_1+(\beta_3-\beta_1)} \, q_1^d q_3 \\
&+ \sum_d < \beta_{10}, \beta_{11}, \beta_2 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 \beta_{12} \\
&+ \sum_d < \beta_{10}, \beta_{11}, \beta_3 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (\beta_{11} + \beta_{12})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{d\neq 0} <\beta_{10},\beta_{11},\beta_4>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_5)\\
&+\sum_{d\neq 0} <\beta_{10},\beta_{11},\beta_5>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\
=&\beta_{10}\beta_{11}+\int_{\beta_3}\beta_{10}\int_{\beta_3}\beta_{11}\ <pt>_{\beta_3}\ q_1q_3\\
&+\int_{\beta_2}\beta_{10}\int_{\beta_2}\beta_{11}\ <\beta_2>_{\beta_2}\ q_1q_2\beta_{12}\\
&+\int_{\beta_2}\beta_{10}\int_{\beta_2}\beta_{11}\ <\beta_3>_{\beta_2}\ q_1q_2(\beta_{11}+\beta_{12})\\
&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{10}\int_{d\beta_1}\beta_{11}\ <\beta_4>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_5)\\
&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{10}\int_{d\beta_1}\beta_{11}\ <\beta_5>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\
=&\beta_{10}\beta_{11},
\end{aligned}
$$

where we used the fact that $\int_{\beta_1}\beta_{11}=0$.

Also by analogy,

$$
\begin{aligned}
\beta_{10}*\beta_{12}=&\beta_{10}\beta_{12}+\int_{\beta_3}\beta_{10}\int_{\beta_3}\beta_{12}\ <pt>_{\beta_3}\ q_1q_3\\
&+\int_{\beta_2}\beta_{10}\int_{\beta_2}\beta_{12}\ <\beta_2>_{\beta_2}\ q_1q_2\beta_{12}\\
&+\int_{\beta_2}\beta_{10}\int_{\beta_2}\beta_{12}\ <\beta_3>_{\beta_2}\ q_1q_2(\beta_{11}+\beta_{12})\\
&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{10}\int_{d\beta_1}\beta_{12}\ <\beta_4>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_5)\\
&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{10}\int_{d\beta_1}\beta_{12}\ <\beta_5>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\
=&\beta_{10}\beta_{12},
\end{aligned}
$$

where we used the fact that $\int_{\beta_1}\beta_{12}=0$;

$$\begin{aligned}\beta_{11}*\beta_{11}=&\beta_{11}^2+\int_{\beta_3}\beta_{11}\int_{\beta_3}\beta_{11}\ <pt>_{\beta_3}\ q_1q_3\\&+\int_{\beta_2}\beta_{11}\int_{\beta_2}\beta_{11}\ <\beta_2>_{\beta_2}\ q_1q_2\beta_{12}\\&+\int_{\beta_2}\beta_{11}\int_{\beta_2}\beta_{11}\ <\beta_3>_{\beta_2}\ q_1q_2(\beta_{11}+\beta_{12})\\&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{11}\int_{d\beta_1}\beta_{11}\ <\beta_4>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_5)\\&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{11}\int_{d\beta_1}\beta_{11}\ <\beta_5>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\=&\beta_{11}^2+2q_1q_3-q_1q_2\beta_{12}+q_1q_2(\beta_{11}+\beta_{12})\\=&\beta_{11}^2+q_1q_2\beta_{11}+2q_1q_3,\end{aligned}$$

where we used the fact that $\int_{\beta_2}\beta_{11}=-1,\int_{\beta_3}\beta_{11}=1$;

$$\begin{aligned}\beta_{11}*\beta_{12}=&\beta_{11}\beta_{12}+\int_{\beta_3}\beta_{11}\int_{\beta_3}\beta_{12}\ <pt>_{\beta_3}\ q_1q_3\\&+\int_{\beta_2}\beta_{11}\int_{\beta_2}\beta_{12}\ <\beta_2>_{\beta_2}\ q_1q_2\beta_{12}\\&+\int_{\beta_2}\beta_{11}\int_{\beta_2}\beta_{12}\ <\beta_3>_{\beta_2}\ q_1q_2(\beta_{11}+\beta_{12})\\&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{11}\int_{d\beta_1}\beta_{12}\ <\beta_4>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_5)\\&+\sum_{d\neq 0}\int_{d\beta_1}\beta_{11}\int_{d\beta_1}\beta_{12}\ <\beta_5>_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\=&\beta_{11}\beta_{12}+q_1q_2\beta_{12}-q_1q_2(\beta_{11}+\beta_{12})\\=&\beta_{11}\beta_{12}-q_1q_2\beta_{11},\end{aligned}$$

where we used the fact that $\int_{\beta_2} \beta_{12} = -1, \int_{\beta_3} \beta_{12} = 0$; and finally,

$$\begin{aligned}
\beta_{12} * \beta_{12} =& \beta_{12}^2 + \int_{\beta_3} \beta_{12} \int_{\beta_3} \beta_{12} \ < pt >_{\beta_3} q_1 q_3 \\
&+ \int_{\beta_2} \beta_{12} \int_{\beta_2} \beta_{12} \ < \beta_2 >_{\beta_2} q_1 q_2 \beta_{12} \\
&+ \int_{\beta_2} \beta_{12} \int_{\beta_2} \beta_{12} \ < \beta_3 >_{\beta_2} q_1 q_2 (\beta_{11} + \beta_{12}) \\
&+ \sum_{d \neq 0} \int_{d\beta_1} \beta_{12} \int_{d\beta_1} \beta_{12} \ < \beta_4 >_{d\beta_1} q_1^d (-\frac{1}{2}\beta_5) \\
&+ \sum_{d \neq 0} \int_{d\beta_1} \beta_{12} \int_{d\beta_1} \beta_{12} \ < \beta_5 >_{d\beta_1} q_1^d (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
=& \beta_{12}^2 - q_1 q_2 \beta_{12} + q_1 q_2 (\beta_{11} + \beta_{12}) \\
=& \beta_{12}^2 + q_1 q_2 \beta_{11}.
\end{aligned}$$

Now we summarize what we got above:

$$\begin{aligned}
\beta_{10} * \beta_{10} =& \beta_{10}^2 + 8 \sum_{d \neq 0} d q_1^d \beta_4 + 12 \sum_{d \neq 0} d q_1^d \beta_5, \\
\beta_{10} * \beta_{11} =& \beta_{10} \beta_{11}, \\
\beta_{10} * \beta_{12} =& \beta_{10} \beta_{12}, \\
\beta_{11} * \beta_{11} =& \beta_{11}^2 + q_1 q_2 \beta_{11} + 2 q_1 q_3, \\
\beta_{11} * \beta_{12} =& \beta_{11} \beta_{12} - q_1 q_2 \beta_{11}, \\
\beta_{12} * \beta_{12} =& \beta_{12}^2 + q_1 q_2 \beta_{11}.
\end{aligned} \tag{5.1}$$

For later use, we express the basis elements in terms of quantum products of generators in view of the list of intersection products at page 29:

$$\begin{aligned}
&\beta_4 = \frac{1}{2} \beta_{10} * \beta_{11}, \qquad \beta_5 = \frac{1}{2} \beta_{10} * \beta_{12}, \\
&\beta_6 = \beta_{11} * \beta_{11} + \beta_{11} * \beta_{12} - 2 q_1 q_3, \\
&\beta_7 = \beta_{12} * \beta_{12} - q_1 q_2 \beta_{11}, \\
&\beta_8 = \beta_{11} * \beta_{12} + q_1 q_2 \beta_{11} - \beta_9.
\end{aligned} \tag{5.2}$$

Furthermore,

$$\begin{aligned}
\beta_9 * \beta_{10} =& \beta_9\beta_{10} + \sum_d < \beta_9, \beta_{10}, pt >_{d\beta_1+3(\beta_2-\beta_1)} \ q_1^d q_2^3 \\
&+ \sum_d < \beta_9, \beta_{10}, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \ q_1^d q_2 q_3 \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_1 >_{d\beta_1+2(\beta_2-\beta_1)} \ q_1^d q_2^2 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_2 >_{d\beta_1+2(\beta_2-\beta_1)} \ q_1^d q_2^2 \beta_{12} \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_3 >_{d\beta_1+2(\beta_2-\beta_1)} \ q_1^d q_2^2 (\beta_{11} + \beta_{12}) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_1 >_{d\beta_1+(\beta_3-\beta_1)} \ q_1^d q_3 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)} \ q_1^d q_3 \beta_{12} \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_3 >_{d\beta_1+(\beta_3-\beta_1)} \ q_1^d q_3 (\beta_{11} + \beta_{12}) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_4 >_{d\beta_1+(\beta_2-\beta_1)} \ q_1^d q_2 (-\frac{1}{2}\beta_5) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_5 >_{d\beta_1+(\beta_2-\beta_1)} \ q_1^d q_2 (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_6 >_{d\beta_1+(\beta_2-\beta_1)} \ q_1^d q_2 \cdot \frac{1}{2}\beta_7 \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_7 >_{d\beta_1+(\beta_2-\beta_1)} \ q_1^d q_2 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_8 >_{d\beta_1+(\beta_2-\beta_1)} \ q_1^d q_2 (\frac{1}{2}\beta_7 + \beta_8) \\
&+ \sum_d < \beta_9, \beta_{10}, \beta_9 >_{d\beta_1+(\beta_2-\beta_1)} \ q_1^d q_2 \beta_9 \\
&+ \sum_{d\neq 0} < \beta_9, \beta_{10}, \beta_{10} >_{d\beta_1} \ q_1^d (-\frac{1}{2}\beta_1) \\
&+ \sum_{d\neq 0} < \beta_9, \beta_{10}, \beta_{11} >_{d\beta_1} \ q_1^d \beta_3 \\
&+ \sum_{d\neq 0} < \beta_9, \beta_{10}, \beta_{12} >_{d\beta_1} \ q_1^d (\beta_2 + \beta_3)
\end{aligned}$$

$$
\begin{aligned}
=&\beta_9\beta_{10} + \sum_d \int_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \beta_{10} < \beta_9, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} q_1^d q_2 q_3 \\
&+ \sum_d \int_{d\beta_1+(\beta_3-\beta_1)} \beta_{10} < \beta_9, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 \beta_{12} \\
=&\beta_9\beta_{10}.
\end{aligned}
$$

Here we used the fact that $< \beta_9, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}$ and $< \beta_9, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)}$ are nontrivial only when $d = 2$ and $d = 1$ respectively, but $\int_{\beta_2+\beta_3} \beta_{10} = \int_{\beta_3} \beta_{10} = 0$.

Similarly, we substitute β_{11} and β_{12} for β_{10} in the expression of $\beta_9 * \beta_{10}$ respectively, and omit trivial terms to get

$$
\begin{aligned}
\beta_9 * \beta_{11} =&\beta_9\beta_{11} + \sum_d \int_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \beta_{11} < \beta_9, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} q_1^d q_2 q_3 \\
&+ \sum_d \int_{d\beta_1+(\beta_3-\beta_1)} \beta_{11} < \beta_9, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 \beta_{12} \\
=&\beta_9\beta_{11} + q_1 q_3 \beta_{12}, \\
\beta_9 * \beta_{12} =&\beta_9\beta_{12} + \sum_d \int_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \beta_{12} < \beta_9, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} q_1^d q_2 q_3 \\
&+ \sum_d \int_{d\beta_1+(\beta_3-\beta_1)} \beta_{12} < \beta_9, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 \beta_{12} \\
=&\beta_9\beta_{12} + 2q_1^2 q_2 q_3.
\end{aligned}
$$

We list these results as follows:

$$
\begin{aligned}
\beta_9 * \beta_{10} =&\beta_9\beta_{10}, \\
\beta_9 * \beta_{11} =&\beta_9\beta_{11} + q_1 q_3 \beta_{12}, \\
\beta_9 * \beta_{12} =&\beta_9\beta_{12} + 2q_1^2 q_2 q_3.
\end{aligned} \tag{5.3}
$$

The basis elements $\beta_1, \beta_2, \beta_3$ are expressed as

$$
\begin{aligned}
\beta_1 &= \frac{1}{2}\beta_9 * \beta_{10}, \\
\beta_2 &= \beta_9 * \beta_{11} - q_1 q_3 \beta_{12}, \\
\beta_3 &= \beta_9 * \beta_{12} - 2q_1^2 q_2 q_3.
\end{aligned} \tag{5.4}
$$

With all the computational results in §4.5, we can compute

$$
\begin{aligned}
\beta_9 * \beta_9 =& \beta_9^2 + \sum_d < \beta_9, \beta_9, pt >_{d\beta_1+4(\beta_2-\beta_1)} \; q_1^d q_2^4 \\
&+ \sum_d < \beta_9, \beta_9, pt >_{d\beta_1+2(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2^2 q_3 \\
&+ \sum_d < \beta_9, \beta_9, pt >_{d\beta_1+2(\beta_3-\beta_1)} \; q_1^d q_3^2 \\
&+ \sum_d < \beta_9, \beta_9, \beta_1 >_{d\beta_1+3(\beta_2-\beta_1)} \; q_1^d q_2^3 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_9, \beta_9, \beta_2 >_{d\beta_1+3(\beta_2-\beta_1)} \; q_1^d q_2^3 \beta_{12} \\
&+ \sum_d < \beta_9, \beta_9, \beta_3 >_{d\beta_1+3(\beta_2-\beta_1)} \; q_1^d q_2^3 (\beta_{11} + \beta_{12}) \\
&+ \sum_d < \beta_9, \beta_9, \beta_1 >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2 q_3 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_9, \beta_9, \beta_2 >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2 q_3 \beta_{12} \\
&+ \sum_d < \beta_9, \beta_9, \beta_3 >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2 q_3 (\beta_{11} + \beta_{12}) \\
&+ \sum_d < \beta_9, \beta_9, \beta_4 >_{d\beta_1+2(\beta_2-\beta_1)} \; q_1^d q_2^2 (-\frac{1}{2}\beta_5) \\
&+ \sum_d < \beta_9, \beta_9, \beta_5 >_{d\beta_1+2(\beta_2-\beta_1)} \; q_1^d q_2^2 (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
&+ \sum_d < \beta_9, \beta_9, \beta_6 >_{d\beta_1+2(\beta_2-\beta_1)} \; q_1^d q_2^2 (\frac{1}{2}\beta_7) \\
&+ \sum_d < \beta_9, \beta_9, \beta_7 >_{d\beta_1+2(\beta_2-\beta_1)} \; q_1^d q_2^2 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
&+ \sum_d < \beta_9, \beta_9, \beta_8 >_{d\beta_1+2(\beta_2-\beta_1)} \; q_1^d q_2^2 (\frac{1}{2}\beta_7 + \beta_8)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_d <\beta_9,\beta_9,\beta_9>_{d\beta_1+2(\beta_2-\beta_1)}\ q_1^d q_2^2 \beta_9 \\
&+\sum_d <\beta_9,\beta_9,\beta_4>_{d\beta_1+(\beta_3-\beta_1)}\ q_1^d q_3(-\frac{1}{2}\beta_5) \\
&+\sum_d <\beta_9,\beta_9,\beta_5>_{d\beta_1+(\beta_3-\beta_1)}\ q_1^d q_3(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5) \\
&+\sum_d <\beta_9,\beta_9,\beta_6>_{d\beta_1+(\beta_3-\beta_1)}\ q_1^d q_3(\frac{1}{2}\beta_7) \\
&+\sum_d <\beta_9,\beta_9,\beta_7>_{d\beta_1+(\beta_3-\beta_1)}\ q_1^d q_3(\frac{1}{2}\beta_6+\frac{1}{2}\beta_7+\frac{1}{2}\beta_8) \\
&+\sum_d <\beta_9,\beta_9,\beta_8>_{d\beta_1+(\beta_3-\beta_1)}\ q_1^d q_3(\frac{1}{2}\beta_7+\beta_8) \\
&+\sum_d <\beta_9,\beta_9,\beta_9>_{d\beta_1+(\beta_3-\beta_1)}\ q_1^d q_3\beta_9 \\
&+\sum_d <\beta_9,\beta_9,\beta_{10}>_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(-\frac{1}{2}\beta_1) \\
&+\sum_d <\beta_9,\beta_9,\beta_{11}>_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2\beta_3 \\
&+\sum_d <\beta_9,\beta_9,\beta_{12}>_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(\beta_2+\beta_3) \\
=&\beta_9^2+\sum_d <\beta_9,\beta_9,\beta_3>_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^d q_2 q_3(\beta_{11}+\beta_{12}) \\
=&\beta_9^2+q_1^2q_2q_3(\beta_{11}+\beta_{12}) \\
=&\beta_9^2+q_1^2q_2q_3\beta_{11}+q_1^2q_2q_3\beta_{12}.
\end{aligned}
$$

We rewrite the result of this computation as

$$\beta_9 * \beta_9 = \beta_9^2 + q_1^2q_2q_3\beta_{11} + q_1^2q_2q_3\beta_{12}. \tag{5.5}$$

5.2 Associativity of Quantum Product

Gromov-Witten invariants enjoy strong relations arising from the associativity of the quantum product. In this section, we make use of the associativity of the quantum product to derive some relations of Gromov-Witten invariants and then simplify these relations to compute the invariants. This finishes the project we initiated in the previous chapter to compute all the two-pointed invariants. In the next section, we'll apply these invariants to the computations of quantum products, thus determining the quantum product ring in the last section.

First, we study the associativity identity

$$\beta_{10} * (\beta_{10} * \beta_{12}) = (\beta_{10} * \beta_{10}) * \beta_{12}.$$

From the equalities in $(5.1, 5.2)$ in §5.1, the left-hand side is equal to

$$\beta_{10} * (\beta_{10}\beta_{12}) = \beta_{10} * (2\beta_5) = 2\beta_5 * \beta_{10},$$

the right-hand side is equal to

$$\begin{aligned}
&(\beta_{10}^2 + 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5) * \beta_{12} \\
=&(4\beta_4 + 6\beta_5 - 2\beta_7 - 4\beta_8 - 4\beta_9 + 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5) * \beta_{12} \\
=&4\beta_4 * \beta_{12} + 6\beta_5 * \beta_{12} - 2\beta_7 * \beta_{12} - 4\beta_8 * \beta_{12} - 4\beta_9 * \beta_{12} \\
&\quad + 8\sum_{d\neq 0} dq_1^d\beta_4 * \beta_{12} + 12\sum_{d\neq 0} dq_1^d\beta_5 * \beta_{12},
\end{aligned}$$

recalling that

$$\beta_{10}^2 = 4\beta_4 + 6\beta_5 - 2\beta_7 - 4\beta_8 - 4\beta_9.$$

Then equating the two sides, we obtain the first identity

$$\begin{aligned}
&2\beta_4 * \beta_{12} + 3\beta_5 * \beta_{12} - \beta_7 * \beta_{12} - 2\beta_8 * \beta_{12} - 2\beta_9 * \beta_{12} \\
&\quad + 4\sum_{d\neq 0} dq_1^d\beta_4 * \beta_{12} + 6\sum_{d\neq 0} dq_1^d\beta_5 * \beta_{12} = \beta_5 * \beta_{10}
\end{aligned} \tag{5.6}$$

Then we look at the associativity identity

$$\beta_{10} * (\beta_{12} * \beta_{12}) = (\beta_{10} * \beta_{12}) * \beta_{12}.$$

The left-hand side is equal to

$$\beta_{10} * (\beta_{12}^2 + q_1 q_2 \beta_{11}) = \beta_{10} * \beta_7 + q_1 q_2 \beta_{10} * \beta_{11}$$
$$= \beta_7 * \beta_{10} + q_1 q_2 \beta_{10} \beta_{11} = \beta_7 * \beta_{10} + 2 q_1 q_2 \beta_4,$$

the right-hand side is equal to

$$(\beta_{10}\beta_{12}) * \beta_{12} = 2\beta_5 * \beta_{12}.$$

Then we get the second identity

$$\beta_7 * \beta_{10} + 2 q_1 q_2 \beta_4 = 2\beta_5 * \beta_{12} \tag{5.7}$$

Finally, we look at the associativity identity

$$\beta_{10} * (\beta_{11} * \beta_{12}) = (\beta_{10} * \beta_{11}) * \beta_{12}.$$

The left-hand side is equal to

$$\beta_{10} * (\beta_{11}\beta_{12} - q_1 q_2 \beta_{11}) = \beta_{10} * (\beta_8 + \beta_9) - q_1 q_2 \beta_{10} * \beta_{11}$$
$$= \beta_{10} * (\beta_8 + \beta_9) - q_1 q_2 \beta_{10}\beta_{11} = \beta_8 * \beta_{10} + \beta_9 * \beta_{10} - 2 q_1 q_2 \beta_4$$

the right-hand side is equal to

$$(\beta_{10}\beta_{11}) * \beta_{12} = 2\beta_4 * \beta_{12},$$

so we get the third identity

$$\beta_8 * \beta_{10} + \beta_9 * \beta_{10} - 2 q_1 q_2 \beta_4 = 2\beta_4 * \beta_{12} \tag{5.8}$$

Also

$$(\beta_{10} * \beta_{11}) * \beta_{12} = \beta_{11} * (\beta_{10} * \beta_{12}),$$

and the right-hand side is equal to

$$\beta_{11} * (\beta_{10}\beta_{12}) = \beta_{11} * (2\beta_5) = 2\beta_5 * \beta_{11},$$

and thus we get the fourth equation

$$\beta_4 * \beta_{12} = \beta_5 * \beta_{11} \tag{5.9}$$

For each identity, as we compute the quantum products, we can compare and equate the terms at the two sides corresponding to the same cohomology class and the same degrees of powers of the parameters. We first consider the terms corresponding to the cohomology class 1 and $q_1^d q_2 q_3$.

Beginning with the second identity (5.7), we get the equation

$$< \beta_7, \beta_{10}, pt >_d \; q_1^d q_2 q_3 = 2 < \beta_5, \beta_{12}, pt >_d \; q_1^d q_2 q_3,$$

where $< \beta_5, \beta_{12}, pt >_d$, etc. means the invariants at the curve class $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$. But

$$< \beta_7, \beta_{10}, pt >_d = \int_\beta \beta_{10} < \beta_7, pt >_d = -2(d-2) < \beta_7, pt >_d,$$
$$< \beta_5, \beta_{12}, pt >_d = \int_\beta \beta_{12} < \beta_5, pt >_d = < \beta_5, pt >_d,$$

so we know for any d,

$$(d-2) < \beta_7, pt >_d = - < \beta_5, pt >_d .$$

From this when $d \neq 2$,

$$< \beta_7, pt >_d = -\frac{1}{d-2} < \beta_5, pt >_d .$$

From the first identity (5.6) above, we get

$$\begin{aligned}
&2\sum_d < \beta_4, \beta_{12}, pt >_d q_1^d q_2 q_3 + 3\sum_d < \beta_5, \beta_{12}, pt >_d q_1^d q_2 q_3 \\
&-\sum_d < \beta_7, \beta_{12}, pt >_d q_1^d q_2 q_3 - 2\sum_d < \beta_8, \beta_{12}, pt >_d q_1^d q_2 q_3 \\
&-2\sum_d < \beta_9, \beta_{12}, pt >_d q_1^d q_2 q_3 + 4\sum_{l\neq 0} l q_1^l \sum_k < \beta_4, \beta_{12}, pt >_k q_1^k q_2 q_3 \\
&+6\sum_{l\neq 0} l q_1^l \sum_k < \beta_5, \beta_{12}, pt >_k q_1^k q_2 q_3 \\
=&\sum_d < \beta_5, \beta_{10}, pt >_d q_1^d q_2 q_3,
\end{aligned}$$

But

$$\begin{aligned}
< \beta_4, \beta_{12}, pt >_d &= \int_\beta \beta_{12} < \beta_4, pt >_d = 1 \cdot 0 = 0, \\
< \beta_8, \beta_{12}, pt >_d &= \int_\beta \beta_{12} < \beta_8, pt >_d = 1 \cdot 0 = 0,
\end{aligned}$$

for any d and for any $d \neq 2$,

$$< \beta_9, \beta_{12}, pt >_d = \int_\beta \beta_{12} < \beta_9, pt >_d = < \beta_9, pt >_d = 0,$$

but when $d = 2, < \beta_9, \beta_{12}, pt >_d = 2$. With these in place, the above equation simplifies to

$$\begin{aligned}
&3\sum_d < \beta_5, pt >_d q_1^d q_2 q_3 - \sum_d < \beta_7, pt >_d q_1^d q_2 q_3 \\
&+6\sum_{l\neq 0} l q_1^l \sum_k < \beta_5, pt >_k q_1^k q_2 q_3 - 4 q_1^2 q_2 q_3 \\
=&-2\sum_d (d-2) < \beta_5, pt >_d q_1^d q_2 q_3,
\end{aligned}$$

or

$$\sum_d (2d-1) < \beta_5, pt >_d q_1^d q_2 q_3 - \sum_d < \beta_7, pt >_d q_1^d q_2 q_3$$
$$+ 6 \sum_{l \neq 0} l q_1^l \sum_k < \beta_5, pt >_k q_1^k q_2 q_3 - 4 q_1^2 q_2 q_3 = 0.$$

Let $a_d =< \beta_5, pt >_d$, then

$$\sum_{l \neq 0} l q_1^l \sum_k < \beta_5, pt >_k q_1^k q_2 q_3 = \sum_d (a_{d-1} + 2a_{d-2} + \cdots + d a_0) q_1^d q_2 q_3.$$

Substituting this in the above equation and equating the terms in front of the power $q_1^d q_2 q_3$, with $d > 2$, we obtain

$$(2d - 1 + \frac{1}{d-2}) a_d + 6(a_{d-1} + 2a_{d-2} + \cdots + d a_0) = 0$$

or

$$a_d = -\frac{6(d-2)}{(2d-3)(d-1)} (a_{d-1} + 2a_{d-2} + \cdots + d a_0)$$

The initial data can be determined directly using localization, by which we get

$$< \beta_5, pt >_0 = 0, < \beta_5, pt >_1 = 1, < \beta_5, pt >_2 = 0.$$

Note that $< \beta_5, pt >_2 = 0$ is compatible with the formula $< \beta_5, pt >_d = -(d-2) < \beta_7, pt >_d$ if we plug in $d = 2$. Then putting these initial values back into the recursive relation above, we get

$$< \beta_5, pt >_d = -\frac{6(d-2)}{(2d-3)(d-1)} (a_{d-1} + 2a_{d-2} + \cdots + (d-3) a_3) - \frac{6(d-2)}{2d-3}$$

and hence $<\beta_7, pt>_d$ can also be determined as

$$\begin{aligned}
<\beta_7, pt>_d &= -\frac{1}{d-2}a_d \\
&= \frac{6}{(2d-3)(d-1)}(a_{d-1}+2a_{d-2}+\cdots+(d-3)a_3)+\frac{6}{2d-3}
\end{aligned}$$

for all $d>2$. But the initial values of $<\beta_7, pt>_d$ are evaluated as

$$<\beta_7, pt>_0 = 0, <\beta_7, pt>_1 = 1, <\beta_7, pt>_2 = 2.$$

Next we consider the terms corresponding to the second degree cohomology classes and powers $q_1^d q_2$ from the associative identities. First we equate the two sides of the identity (5.6) with the class β_4:

$$\begin{aligned}
&2\sum_d <\beta_4, \beta_{12}, \beta_4>_d q_1^d q_2 + 3\sum_d <\beta_5, \beta_{12}, \beta_4>_d q_1^d q_2 \\
&-\sum_d <\beta_7, \beta_{12}, \beta_4>_d q_1^d q_2 - 2\sum_d <\beta_8, \beta_{12}, \beta_4>_d q_1^d q_2 \\
&-2\sum_d <\beta_9, \beta_{12}, \beta_4>_d q_1^d q_2 + 4\sum_{l\neq 0} l q_1^l \sum_k <\beta_4, \beta_{12}, \beta_4>_k q_1^k q_2 \\
&+6\sum_{l\neq 0} l q_1^l \sum_k <\beta_5, \beta_{12}, \beta_4>_k q_1^k q_2 \\
=&\sum_d <\beta_5, \beta_{10}, \beta_4>_d q_1^d q_2.
\end{aligned}$$

With $\beta = d\beta_1 + (\beta_2 - \beta_1)$,

$$\int_\beta \beta_{12} = 1, \int_\beta \beta_{10} = -2(d-1),$$

and

$$<\beta_9, \beta_{12}, \beta_4>_d = <\beta_9, \beta_4>_d = 0,$$

so we can simplify the expression above as

$$\begin{aligned}
&2\sum_d < \beta_4, \beta_4 >_d q_1^d q_2 + \sum_d (2d+1) < \beta_5, \beta_4 >_d q_1^d q_2 \\
&- \sum_d < \beta_7, \beta_4 >_d q_1^d q_2 - 2\sum_d < \beta_8, \beta_4 >_d q_1^d q_2 \\
&+ 4\sum_{l\neq 0} l q_1^l \sum_k < \beta_4, \beta_4 >_k q_1^k q_2 + 6\sum_{l\neq 0} l q_1^l \sum_k < \beta_5, \beta_4 >_k q_1^k q_2 = 0.
\end{aligned} \tag{5.10}$$

From the identity (5.9), we get

$$\sum_d < \beta_4, \beta_{12}, \beta_4 >_d q_1^d q_2 = \sum_d < \beta_5, \beta_{11}, \beta_4 >_d q_1^d q_2$$

so for any d,

$$< \beta_5, \beta_4 >_d = - < \beta_4, \beta_4 >_d .$$

This relation coincides with the one we derived after Proposition 4.4.9.

From the equation (5.7), noting that

$$\beta_4 = 2(-\frac{1}{2}\beta_5) - 2(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5)$$

we have

$$\sum_d < \beta_7, \beta_{10}, \beta_4 >_d q_1^d q_2 + 4q_1q_2 = 2\sum_d < \beta_5, \beta_{12}, \beta_4 >_d q_1^d q_2$$

or

$$-\sum_d (d-1) < \beta_7, \beta_4 >_d q_1^d q_2 + 2q_1q_2 = \sum_d < \beta_5, \beta_4 >_d q_1^d q_2,$$

from which we learn that when $d > 1$,

$$(d-1) < \beta_7, \beta_4 >_d = - < \beta_5, \beta_4 >_d = < \beta_4, \beta_4 >_d$$

or

$$< \beta_7, \beta_4 >_d = \frac{1}{d-1} < \beta_4, \beta_4 >_d .$$

From the equation (5.8), we get

$$\begin{aligned} &\sum_d < \beta_8, \beta_{10}, \beta_4 >_d q_1^d q_2 + \sum_d < \beta_9, \beta_{10}, \beta_4 >_d q_1^d q_2 - 4q_1 q_2 \\ &\quad = 2 \sum_d < \beta_4, \beta_{12}, \beta_4 >_d q_1^d q_2 \end{aligned}$$

or

$$\sum_d (d-1) < \beta_8, \beta_4 >_d q_1^d q_2 + 2q_1 q_2 = -\sum_d < \beta_4, \beta_4 >_d q_1^d q_2.$$

From this we know that when $d > 1$,

$$(d-1) < \beta_8, \beta_4 >_d = - < \beta_4, \beta_4 >_d,$$

or

$$< \beta_8, \beta_4 >_d = -\frac{1}{d-1} < \beta_4, \beta_4 >_d .$$

Substituting all these in the equation (5.10), we obtain an equation for $< \beta_4, \beta_4 >_d$. Let $b_d =< \beta_4, \beta_4 >_d$. When $d > 1$, it is simplified as

$$(1 - 2d + \frac{1}{d-1}) b_d = 2(b_{d-1} + 2b_{d-2} + \cdots + d b_0)$$

or

$$< \beta_4, \beta_4 >_d = b_d = -\frac{2(d-1)}{d(2d-3)} (b_{d-1} + 2b_{d-2} + \cdots + d b_0).$$

The initial values are computed by localization as

$$< \beta_4, \beta_4 >_0 = 1, \quad < \beta_4, \beta_4 >_1 = -2.$$

So we obtain the expressions of the invariants

$$< \beta_4, \beta_5 >_d = -b_d$$

for any d and

$$< \beta_4, \beta_7 >_d = \frac{1}{d-1} b_d, \quad < \beta_4, \beta_8 >_d = -\frac{1}{d-1} b_d,$$

for $d > 1$. Their initial values by localization are

$$< \beta_4, \beta_7 >_0 = -1, \quad < \beta_4, \beta_8 >_0 = 1,$$
$$< \beta_4, \beta_7 >_1 = 0, \quad < \beta_4, \beta_8 >_1 = 0.$$

In the same vain from the associativity identity (5.9), we get for any d,

$$< \beta_4, \beta_{12}, \beta_5 >_d = < \beta_5, \beta_{11}, \beta_5 >_d,$$
$$< \beta_4, \beta_{12}, \beta_7 >_d = < \beta_5, \beta_{11}, \beta_7 >_d,$$
$$< \beta_4, \beta_{12}, \beta_8 >_d = < \beta_5, \beta_{11}, \beta_8 >_d,$$

or

$$< \beta_4, \beta_5 >_d = - < \beta_5, \beta_5 >_d, \qquad < \beta_4, \beta_7 >_d = - < \beta_5, \beta_7 >_d,$$
$$< \beta_4, \beta_8 >_d = - < \beta_5, \beta_8 >_d,$$

and hence for any d,

$$< \beta_5, \beta_5 >_d = b_d$$

and when $d > 1$,

$$< \beta_5, \beta_7 >_d = -\frac{1}{d-1} b_d, \qquad < \beta_5, \beta_8 >_d = \frac{1}{d-1} b_d.$$

Their initial values are

$$< \beta_5, \beta_7 >_0 = 1, \qquad < \beta_5, \beta_8 >_0 = -1,$$
$$< \beta_5, \beta_7 >_1 = 0, \qquad < \beta_5, \beta_8 >_1 = 0.$$

Now making use of the associativity identity (5.7), we get, for any d,

$$< \beta_7, \beta_{10}, \beta_7 >_d = 2 < \beta_5, \beta_{12}, \beta_7 >_d,$$
$$< \beta_7, \beta_{10}, \beta_8 >_d = 2 < \beta_5, \beta_{12}, \beta_8 >_d,$$

or

$$-(d-1) < \beta_7, \beta_7 >_d = < \beta_5, \beta_7 >_d,$$
$$-(d-1) < \beta_7, \beta_8 >_d = < \beta_5, \beta_8 >_d,$$

so when $d > 1$,

$$< \beta_7, \beta_7 >_d = \frac{1}{(d-1)^2} b_d, \qquad < \beta_7, \beta_8 >_d = -\frac{1}{(d-1)^2} b_d.$$

Their initial values are

$$< \beta_7, \beta_7 >_0 = 1, \qquad < \beta_7, \beta_8 >_0 = -1,$$
$$< \beta_7, \beta_7 >_1 = 2, \qquad < \beta_7, \beta_8 >_1 = 0.$$

From the associativity identity (5.8), we get

$$< \beta_8, \beta_{10}, \beta_8 >_d + < \beta_9, \beta_{10}, \beta_8 >_d = 2 < \beta_4, \beta_{12}, \beta_8 >_d,$$

or for any d,

$$-(d-1) < \beta_8, \beta_8 >_d = < \beta_4, \beta_8 >_d,$$

so for $d > 1$,

$$< \beta_8, \beta_8 >_d = \frac{1}{(d-1)^2} b_d.$$

Its initial values are

$$< \beta_8, \beta_8 >_0 = 1, \qquad < \beta_8, \beta_8 >_1 = -1,$$

Till this point, the two-pointed Gromov-Witten invariants with two insertions degree 2 partially computed in Proposition 4.4.9 are completely determined.

In order to finish the computation in part (ii) in Proposition 4.4.5, we continue to work on the associativity law of quantum product, making use of the equalities in (5.3, 5.4). First, we look at

$$\beta_9 * (\beta_{10} * \beta_{11}) = (\beta_9 * \beta_{10}) * \beta_{11} = (\beta_9 * \beta_{11}) * \beta_{10}.$$

By the computational results before, these sides are, respectively,

$$\begin{aligned}
\beta_9 * (\beta_{10} * \beta_{11}) &= \beta_9 * (\beta_{10}\beta_{11}) = \beta_9 * (2\beta_4) = 2\beta_4 * \beta_9, \\
(\beta_9 * \beta_{10}) * \beta_{11} &= (\beta_9\beta_{10}) * \beta_{11} = 2\beta_1 * \beta_{11}, \\
(\beta_9 * \beta_{11}) * \beta_{10} &= (\beta_9\beta_{11} + q_1 q_3 \beta_{12}) * \beta_{10} = \beta_2 * \beta_{10} + q_1 q_3 \beta_{10}\beta_{12} \\
&= \beta_2 * \beta_{10} + 2q_1 q_3 \beta_5,
\end{aligned}$$

so we have the identity

$$2\beta_4 * \beta_9 = 2\beta_1 * \beta_{11} = \beta_2 * \beta_{10} + 2q_1q_3\beta_5. \tag{5.11}$$

Now we look at

$$\beta_9 * (\beta_{10} * \beta_{12}) = \beta_{10} * (\beta_9 * \beta_{12}) = (\beta_{10} * \beta_9) * \beta_{12}.$$

The three sides are equal to

$$\begin{aligned}
&\beta_9 * (\beta_{10} * \beta_{12}) = \beta_9 * (\beta_{10}\beta_{12}) = 2\beta_5 * \beta_9,\\
&\beta_{10} * (\beta_9 * \beta_{12}) = \beta_{10} * (\beta_9\beta_{12} + 2q_1^2q_2q_3) = \beta_3 * \beta_{10} + 2q_1^2q_2q_3\beta_{10},\\
&(\beta_{10} * \beta_9) * \beta_{12} = (\beta_9\beta_{10}) * \beta_{12} = 2\beta_1 * \beta_{12},
\end{aligned}$$

respectively, so we have the identity

$$2\beta_5 * \beta_9 = 2\beta_1 * \beta_{12} = \beta_3 * \beta_{10} + 2q_1^2q_2q_3\beta_{10}. \tag{5.12}$$

Then we look at

$$\beta_9 * (\beta_{11} * \beta_{12}) = \beta_{11} * (\beta_9 * \beta_{12}) = (\beta_{11} * \beta_9) * \beta_{12}.$$

The three sides are respectively equal to

$$\begin{aligned}
\beta_9 * (\beta_{11} * \beta_{12}) &= \beta_9 * (\beta_{11}\beta_{12} - q_1q_2\beta_{11}) = \beta_9 * (\beta_8 + \beta_9) - q_1q_2\beta_9 * \beta_{11}\\
&= \beta_9 * (\beta_8 + \beta_9) - q_1q_2(\beta_9\beta_{11} + q_1q_3\beta_{12})\\
&= \beta_9 * (\beta_8 + \beta_9) - q_1q_2\beta_2 - q_1^2q_2q_3\beta_{12},\\
\beta_{11} * (\beta_9 * \beta_{12}) &= \beta_{11} * (\beta_9\beta_{12} + 2q_1^2q_2q_3) = \beta_3 * \beta_{11} + 2q_1^2q_2q_3\beta_{11},\\
(\beta_{11} * \beta_9) * \beta_{12} &= (\beta_2 + q_1q_3\beta_{12}) * \beta_{12} = \beta_2 * \beta_{12} + q_1q_3(\beta_{12}^2 + q_1q_2\beta_{11})\\
&= \beta_2 * \beta_{12} + q_1q_3\beta_7 + q_1^2q_2q_3\beta_{11},
\end{aligned}$$

so we have the identities

$$\begin{aligned}\beta_9 * (\beta_8 + \beta_9) - q_1q_2\beta_2 - q_1^2q_2q_3\beta_{12} &= \beta_3 * \beta_{11} + 2q_1^2q_2q_3\beta_{11} \\ &= \beta_2 * \beta_{12} + q_1q_3\beta_7 + q_1^2q_2q_3\beta_{11}.\end{aligned} \tag{5.13}$$

Now we consider

$$\beta_9 * (\beta_{12} * \beta_{12}) = (\beta_9 * \beta_{12}) * \beta_{12}.$$

The two sides are equal to

$$\begin{aligned}\beta_9 * (\beta_{12} * \beta_{12}) &= \beta_9 * (\beta_{12}^2 + q_1q_2\beta_{11}) = \beta_7 * \beta_9 + q_1q_2\beta_9 * \beta_{11} \\ &= \beta_7 * \beta_9 + q_1q_2(\beta_9\beta_{11} + q_1q_3\beta_{12}) \\ &= \beta_7 * \beta_9 + q_1q_2\beta_2 + q_1^2q_2q_3\beta_{12}, \\ (\beta_9 * \beta_{12}) * \beta_{12} &= (\beta_9\beta_{12} + 2q_1^2q_2q_3) * \beta_{12} = \beta_3 * \beta_{12} + 2q_1^2q_2q_3\beta_{12},\end{aligned}$$

so we have the identity

$$\beta_7 * \beta_9 + q_1q_2\beta_2 = \beta_3 * \beta_{12} + q_1^2q_2q_3\beta_{12}. \tag{5.14}$$

Next we consider

$$\beta_9 * (\beta_{11} * \beta_{11}) = (\beta_9 * \beta_{11}) * \beta_{11}.$$

The two sides are equal to

$$\begin{aligned}\beta_9 * (\beta_{11} * \beta_{11}) &= \beta_9 * (\beta_{11}^2 + q_1q_2\beta_{11} + 2q_1q_3) \\ &= \beta_9 * (\beta_6 - \beta_8 - \beta_9) + q_1q_2\beta_9 * \beta_{11} + 2q_1q_3\beta_9 \\ &= \beta_6 * \beta_9 - \beta_9 * (\beta_8 + \beta_9) + q_1q_2(\beta_2 + q_1q_3\beta_{12}) + 2q_1q_3\beta_9 \\ &= \beta_6 * \beta_9 - \beta_9 * (\beta_8 + \beta_9) + q_1q_2\beta_2 + q_1^2q_2q_3\beta_{12} + 2q_1q_3\beta_9,\end{aligned}$$

$$
\begin{aligned}
(\beta_9 * \beta_{11}) * \beta_{11} &= (\beta_2 + q_1 q_3 \beta_{12}) * \beta_{11} \\
&= \beta_2 * \beta_{11} + q_1 q_3 \beta_{12} * \beta_{11} \\
&= \beta_2 * \beta_{11} + q_1 q_3 (\beta_{11}\beta_{12} - q_1 q_2 \beta_{11}) \\
&= \beta_2 * \beta_{11} + q_1 q_3 (\beta_8 + \beta_9) - q_1^2 q_2 q_3 \beta_{11}.
\end{aligned}
$$

So we have the identity

$$
\begin{aligned}
&\beta_6 * \beta_9 - \beta_9 * (\beta_8 + \beta_9) + q_1 q_2 \beta_2 + q_1^2 q_2 q_3 \beta_{12} \\
&\quad = \beta_2 * \beta_{11} + q_1 q_3 \beta_8 - q_1 q_3 \beta_9 - q_1^2 q_2 q_3 \beta_{11}.
\end{aligned}
\tag{5.15}
$$

Finally we look at

$$
\beta_9 * (\beta_{10} * \beta_{10}) = (\beta_9 * \beta_{10}) * \beta_{10}.
$$

The right-hand side is equal to

$$
(\beta_9 \beta_{10}) * \beta_{10} = 2\beta_1 * \beta_{10},
$$

the left-hand side is equal to

$$
\begin{aligned}
&(\beta_{10}^2 + 8\sum_{d\neq 0} d q_1^d \beta_4 + 12 \sum_{d\neq 0} d q_1^d \beta_5) * \beta_9 \\
=&(4\beta_4 + 6\beta_5 - 2\beta_7 - 4\beta_8 - 4\beta_9 + 8\sum_{d\neq 0} d q_1^d \beta_4 + 12 \sum_{d\neq 0} d q_1^d \beta_5) * \beta_9 \\
=&4\beta_4 * \beta_9 + 6\beta_5 * \beta_9 - 2\beta_7 * \beta_9 - 4\beta_9 * (\beta_8 + \beta_9) \\
&\quad + 8\sum_{d\neq 0} d q_1^d \beta_4 * \beta_9 + 12 \sum_{d\neq 0} d q_1^d \beta_5 * \beta_9,
\end{aligned}
$$

thus we obtain the identity

$$\begin{aligned}\beta_1 * \beta_{10} =& 2\beta_4 * \beta_9 + 3\beta_5 * \beta_9 - \beta_7 * \beta_9 - 2\beta_9 * (\beta_8 + \beta_9) \\ &+ 4\sum_{d\neq 0} dq_1^d \beta_4 * \beta_9 + 6\sum_{d\neq 0} dq_1^d \beta_5 * \beta_9.\end{aligned} \tag{5.16}$$

From these identities, we collect the following:

$$\begin{aligned}&\beta_4 * \beta_9 = \beta_1 * \beta_{11}, \qquad \beta_5 * \beta_9 = \beta_1 * \beta_{12}, \\ &\beta_7 * \beta_9 = \beta_3 * \beta_{12} - q_1q_2\beta_2 + q_1^2q_2q_3\beta_{12}, \\ &\beta_9 * (\beta_8 + \beta_9) = \beta_3 * \beta_{11} + q_1q_2\beta_2 + 2q_1^2q_2q_3\beta_{11} + q_1^2q_2q_3\beta_{12}.\end{aligned}$$

Substituting all these in the right-hand side of identity (5.16) and simplifying, we have

$$\begin{aligned}&2\beta_1 * \beta_{11} + 3\beta_1 * \beta_{12} - \beta_3 * \beta_{12} - 2\beta_3 * \beta_{11} - q_1q_2\beta_2 - 4q_1^2q_2q_3\beta_{11} \\ &- 3q_1^2q_2q_3\beta_{12} + 4\sum_{d\neq 0} dq_1^d\beta_1 * \beta_{11} + 6\sum_{d\neq 0} dq_1^d\beta_1 * \beta_{12} = \beta_1 * \beta_{10}.\end{aligned} \tag{5.17}$$

Now in this equation, we equate the corresponding terms at the two sides in front of the same cohomology class $-\frac{1}{2}\beta_{10}$ and the same power $q_1^dq_2q_3$ to get

$$\begin{aligned}&2\sum_d < \beta_1, \beta_{11}, \beta_1 >_d q_1^dq_2q_3 + 3\sum_d < \beta_1, \beta_{12}, \beta_1 >_d q_1^dq_2q_3 \\ &- \sum_d < \beta_3, \beta_{12}, \beta_1 >_d q_1^dq_2q_3 - 2\sum_d < \beta_3, \beta_{11}, \beta_1 >_d q_1^dq_2q_3 \\ &+ 4\sum_{l\neq 0} lq_1^l \sum_k < \beta_1, \beta_{11}, \beta_1 >_k q_1^kq_2q_3 + 6\sum_{l\neq 0} lq_1^l \sum_k < \beta_1, \beta_{12}, \beta_1 >_k q_1^kq_2q_3 \\ &= \sum_d < \beta_1, \beta_{10}, \beta_1 >_d q_1^dq_2q_3,\end{aligned}$$

in which $\beta = d\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_1)$. Let $< \beta_1, \beta_1 >_d$ be denoted as c_d. Then from this equation by integrating out β_{11}, β_{12}, we get

$$\begin{aligned}&(2d-1) < \beta_1, \beta_1 >_d - < \beta_3, \beta_1 >_d \\ &+ 6(c_{d-1} + 2c_{d-2} + \cdots + (d-1)c_1 + dc_0) = 0.\end{aligned} \tag{5.18}$$

Carrying out the same process for the identity $2\beta_1 * \beta_{12} = \beta_3 * \beta_{10} + 2q_1^2 q_2 q_3 \beta_{10}$ in (5.12), we obtain

$$2\sum_d < \beta_1, \beta_{12}, \beta_1 >_d q_1^d q_2 q_3 = \sum_d < \beta_3, \beta_{10}, \beta_1 >_d q_1^d q_2 q_3 - 4q_1^2 q_2 q_3$$

From this, we know that for any d,

$$(d-2) < \beta_3, \beta_1 >_d = - < \beta_1, \beta_1 >_d$$

so when $d \neq 2$,

$$< \beta_3, \beta_1 >_d = -\frac{1}{d-2} < \beta_1, \beta_1 >_d .$$

Putting this back into the equation (5.18), we get

$$(2d-1) < \beta_1, \beta_1 >_d + \frac{1}{d-2} < \beta_1, \beta_1 >_d$$
$$+ 6(c_{d-1} + 2c_{d-2} + \cdots + (d-1)c_1 + dc_0) = 0$$

so

$$< \beta_1, \beta_1 >_d = -\frac{6(d-2)}{(d-1)(2d-3)}(c_{d-1} + 2c_{d-2} + \cdots + (d-1)c_1 + dc_0).$$

Its initial values are by localization

$$< \beta_1, \beta_1 >_0 = 0, \quad < \beta_1, \beta_1 >_1 = 1, \quad < \beta_1, \beta_1 >_2 = -2.$$

Obviously, when $d \neq 2$,

$$< \beta_3, \beta_1 >_d = -\frac{1}{d-2} c_d.$$

Its initial values are by localization

$$< \beta_3, \beta_1 >_0 = 0, \quad < \beta_3, \beta_1 >_1 = 1, \quad < \beta_3, \beta_1 >_2 = 0.$$

Again from the identity $2\beta_1 * \beta_{12} = \beta_3 * \beta_{10} + 2q_1^2 q_2 q_3 \beta_{10}$, we get

$$2 < \beta_1, \beta_{12}, \beta_3 >_d = < \beta_3, \beta_{10}, \beta_3 >_d$$

so for any d,

$$< \beta_1, \beta_3 >_d = -(d-2) < \beta_3, \beta_3 >_d .$$

Thus when $d > 2$, we obtain

$$\begin{aligned} < \beta_3, \beta_3 >_d &= \frac{1}{(d-2)^2} c_d \\ &= -\frac{6}{(d-1)(d-2)(2d-3)}(c_{d-1} + 2c_{d-2} + \cdots + (d-1)c_1 + dc_0). \end{aligned}$$

Its initial values are

$$< \beta_3, \beta_3 >_0 = 0, \quad < \beta_3, \beta_3 >_1 = 1, \quad < \beta_3, \beta_3 >_2 = 2.$$

These are the complements to the results of Proposition 4.4.5.

Finally, we equate the terms at two sides in front of classes $-\frac{1}{2}\beta_5$ and $\frac{1}{2}\beta_7$ respectively and power $q_1^d q_3$ in (5.17) to get

$$\begin{aligned} & 2\sum_d < \beta_1, \beta_{11}, \beta_4 >_d q_1^d q_3 + 3\sum_d < \beta_1, \beta_{12}, \beta_4 >_d q_1^d q_3 \\ & - \sum_d < \beta_3, \beta_{12}, \beta_4 >_d q_1^d q_3 - 2\sum_d < \beta_3, \beta_{11}, \beta_4 >_d q_1^d q_3 \\ & + 4\sum_{l\neq 0} l q_1^l \sum_k < \beta_1, \beta_{11}, \beta_4 >_k q_1^k q_3 + 6\sum_{l\neq 0} l q_1^l \sum_k < \beta_1, \beta_{12}, \beta_4 >_k q_1^k q_3 \\ & = \sum_d < \beta_1, \beta_{10}, \beta_4 >_d q_1^d q_3, \end{aligned}$$

and

$$
\begin{aligned}
&2\sum_d <\beta_1,\beta_{11},\beta_6>_d q_1^d q_3 + 3\sum_d <\beta_1,\beta_{12},\beta_6>_d q_1^d q_3\\
&-\sum_d <\beta_3,\beta_{12},\beta_6>_d q_1^d q_3 - 2\sum_d <\beta_3,\beta_{11},\beta_6>_d q_1^d q_3\\
&+4\sum_{l\neq 0} l q_1^l \sum_k <\beta_1,\beta_{11},\beta_6>_k q_1^k q_3 + 6\sum_{l\neq 0} l q_1^l \sum_k <\beta_1,\beta_{12},\beta_6>_k q_1^k q_3\\
&=\sum_d <\beta_1,\beta_{10},\beta_6>_d q_1^d q_3,
\end{aligned}
$$

in which the curve class $\beta = d\beta_1 + (\beta_3 - \beta_1)$.

Simplified, these give

$$
\begin{aligned}
&\sum_d <\beta_1,\beta_4>_d q_1^d q_3 + 2\sum_{l\neq 0} l q_1^l \sum_k <\beta_1,\beta_4>_k q_1^k q_3\\
&= -\sum_d (d-1) <\beta_1,\beta_4>_d q_1^d q_3,\\
&\sum_d <\beta_1,\beta_6>_d q_1^d q_3 + 2\sum_{l\neq 0} l q_1^l \sum_k <\beta_1,\beta_6>_k q_1^k q_3 - 2q_1 q_3\\
&= -\sum_d (d-1) <\beta_1,\beta_6>_d q_1^d q_3,
\end{aligned}
$$

noting that

$$
\int_{d\beta_1+(\beta_3-\beta_1)} \beta_{10} = -2(d-1), \quad \int_{d\beta_1+(\beta_3-\beta_1)} \beta_{11} = 1, \quad \int_{d\beta_1+(\beta_3-\beta_1)} \beta_{12} = 0
$$

and $<\beta_3,\beta_4>_d = 0$ for all d, $<\beta_3,\beta_6>_d = 0$ for $d \neq 1$, 2 for $d = 1$.

Let $f_d = <\beta_1,\beta_4>_d, g_d = <\beta_1,\beta_6>_d$. Then solving these equations, we get for any d,

$$
<\beta_1,\beta_4>_d = -\frac{2}{d}(f_{d-1} + 2f_{d-2} + \cdots + (d-1)f_1 + df_0)
$$

and for $d > 1$,

$$
<\beta_1,\beta_6>_d = -\frac{2}{d}(g_{d-1} + 2g_{d-2} + \cdots + (d-1)g_1 + dg_0).
$$

Their initial values by localization are

$$< \beta_1, \beta_4 >_0 = 1, \qquad < \beta_1, \beta_4 >_1 = -2,$$
$$< \beta_1, \beta_6 >_0 = 1, \qquad < \beta_1, \beta_6 >_1 = 0.$$

From the identity $2\beta_1 * \beta_{11} = \beta_2 * \beta_{10} + 2q_1q_3\beta_5$ in (5.11), we again equate the terms at two sides in front of class $-\frac{1}{2}\beta_5$ and $\frac{1}{2}\beta_7$ respectively to get

$$2\sum_d < \beta_1, \beta_{11}, \beta_4 >_d q_1^d q_3 = \sum_d < \beta_2, \beta_{10}, \beta_4 >_d q_1^d q_3 - 4q_1q_3,$$
$$2\sum_d < \beta_1, \beta_{11}, \beta_6 >_d q_1^d q_3 = \sum_d < \beta_2, \beta_{10}, \beta_6 >_d q_1^d q_3,$$

or

$$\sum_d < \beta_1, \beta_4 >_d q_1^d q_3 = -\sum_d (d-1) < \beta_2, \beta_4 >_d q_1^d q_3 - 2q_1q_3,$$
$$\sum_d < \beta_1, \beta_6 >_d q_1^d q_3 = -\sum_d (d-1) < \beta_2, \beta_6 >_d q_1^d q_3.$$

So for any $d > 1$,

$$< \beta_1, \beta_4 >_d = -(d-1) < \beta_2, \beta_4 >_d,$$
$$< \beta_1, \beta_6 >_d = -(d-1) < \beta_2, \beta_6 >_d,$$

and hence

$$< \beta_2, \beta_4 >_d = -\frac{1}{d-1} f_d = \frac{2}{d(d-1)}(f_{d-1} + 2f_{d-2} + \cdots + (d-1)f_1 + df_0),$$
$$< \beta_2, \beta_6 >_d = -\frac{1}{d-1} g_d = \frac{2}{d(d-1)}(g_{d-1} + 2g_{d-2} + \cdots + (d-1)g_1 + dg_0).$$

Their initial values are

$$< \beta_2, \beta_4 >_0 = 1, \qquad < \beta_2, \beta_4 >_1 = 0,$$

$$< \beta_2, \beta_6 >_0 = 1, \qquad < \beta_2, \beta_6 >_1 = 1.$$

Thus the computation in Proposition 4.4.7 is completed.

5.3 Quantum Products Continued

We aim at deciding the quantum product ring structure of the Hilbert scheme $F^{[2]}$. For this purpose, results of quantum products of basis elements are needed. We have worked out some products in the first section of this chapter, but in order to carry this project through, quantum products of more elements should be put into play. In this section, we solve this problem, making use of the results in the previous section.

We start with $\beta_7 * \beta_{10}$. By definition,

$$\begin{aligned}
\beta_7 * \beta_{10} =& \beta_7\beta_{10} + \sum_d < \beta_7, \beta_{10}, pt >_{d\beta_1+3(\beta_2-\beta_1)} q_1^d q_2^3 \\
&+ \sum_d < \beta_7, \beta_{10}, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} q_1^d q_2 q_3 \\
&+ \sum_d < \beta_7, \beta_{10}, \beta_1 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_7, \beta_{10}, \beta_2 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 \beta_{12} \\
&+ \sum_d < \beta_7, \beta_{10}, \beta_3 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 (\beta_{11} + \beta_{12}) \\
&+ \sum_d < \beta_7, \beta_{10}, \beta_1 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_7, \beta_{10}, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 \beta_{12} \\
&+ \sum_d < \beta_7, \beta_{10}, \beta_3 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 (\beta_{11} + \beta_{12}) \\
&+ \sum_d < \beta_7, \beta_{10}, \beta_4 >_{d\beta_1+(\beta_2-\beta_1)} q_1^d q_2 (-\frac{1}{2}\beta_5)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{d} < \beta_7,\beta_{10},\beta_5 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\
&+\sum_{d} < \beta_7,\beta_{10},\beta_6 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2\cdot\frac{1}{2}\beta_7\\
&+\sum_{d} < \beta_7,\beta_{10},\beta_7 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(\frac{1}{2}\beta_6+\frac{1}{2}\beta_7+\frac{1}{2}\beta_8)\\
&+\sum_{d} < \beta_7,\beta_{10},\beta_8 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(\frac{1}{2}\beta_7+\beta_8)\\
&+\sum_{d} < \beta_7,\beta_{10},\beta_9 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2\beta_9+\sum_{d\neq 0} < \beta_7,\beta_{10},\beta_{10} >_{d\beta_1}\ q_1^d(-\frac{1}{2}\beta_1)\\
&+\sum_{d\neq 0} < \beta_7,\beta_{10},\beta_{11} >_{d\beta_1}\ q_1^d\beta_3+\sum_{d\neq 0} < \beta_7,\beta_{10},\beta_{12} >_{d\beta_1}\ q_1^d(\beta_2+\beta_3)\\
=&\beta_7\beta_{10}+\sum_{d}\int_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\beta_{10} < \beta_7, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^d q_2 q_3\\
&+\sum_{d}\int_{d\beta_1+(\beta_2-\beta_1)}\beta_{10} < \beta_7,\beta_4 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(-\frac{1}{2}\beta_5)\\
&+\sum_{d}\int_{d\beta_1+(\beta_2-\beta_1)}\beta_{10} < \beta_7,\beta_5 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\
&+\sum_{d}\int_{d\beta_1+(\beta_2-\beta_1)}\beta_{10} < \beta_7,\beta_6 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2\cdot\frac{1}{2}\beta_7\\
&+\sum_{d}\int_{d\beta_1+(\beta_2-\beta_1)}\beta_{10} < \beta_7,\beta_7 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(\frac{1}{2}\beta_6+\frac{1}{2}\beta_7+\frac{1}{2}\beta_8)\\
&+\sum_{d}\int_{d\beta_1+(\beta_2-\beta_1)}\beta_{10} < \beta_7,\beta_8 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(\frac{1}{2}\beta_7+\beta_8)\\
=&\beta_7\beta_{10}-2\sum_{d}(d-2) < \beta_7, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^d q_2 q_3\\
&-2\sum_{d}(d-1) < \beta_7,\beta_4 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(-\frac{1}{2}\beta_5)\\
&-2\sum_{d}(d-1) < \beta_7,\beta_5 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\
&-2\sum_{d}(d-1) < \beta_7,\beta_6 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2\cdot\frac{1}{2}\beta_7\\
&-2\sum_{d}(d-1) < \beta_7,\beta_7 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(\frac{1}{2}\beta_6+\frac{1}{2}\beta_7+\frac{1}{2}\beta_8)\\
&-2\sum_{d}(d-1) < \beta_7,\beta_8 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^d q_2(\frac{1}{2}\beta_7+\beta_8)
\end{aligned}
$$

$$
\begin{aligned}
=&\beta_7\beta_{10} + 2\sum_{d} a_d q_1^d q_2 q_3 - 2\sum_{d\neq 1} b_d q_1^d q_2(-\frac{1}{2}\beta_5)\\
&+ 2\sum_{d\neq 1} b_d q_1^d q_2(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) - 2\sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2(\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8)\\
&+ 2\sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2(\frac{1}{2}\beta_7 + \beta_8)\\
=&\beta_7\beta_{10} + 2\sum_{d} a_d q_1^d q_2 q_3 - \sum_{d\neq 1} b_d q_1^d q_2\beta_4 - \sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2\beta_6 + \sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2\beta_8\\
=&\beta_7\beta_{10} + 2\sum_{d} a_d q_1^d q_2 q_3 - \frac{1}{2}\sum_{d\neq 1} b_d q_1^d q_2\beta_{10} * \beta_{11}\\
&- \sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2(\beta_{11} * \beta_{11} + \beta_{11} * \beta_{12} - 2q_1q_3)\\
&+ \sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2(\beta_{11} * \beta_{12} + q_1q_2\beta_{11} - \beta_9)\\
=&\beta_7\beta_{10} - \frac{1}{2}\sum_{d\neq 1} b_d q_1^d q_2\beta_{10} * \beta_{11} - \sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2\beta_{11} * \beta_{11} + \sum_{d\neq 1}\frac{b_d}{d-1} q_1^{d+1} q_2^2\beta_{11}\\
&- \sum_{d\neq 1}\frac{b_d}{d-1} q_1^d q_2\beta_9 + 2\sum_{d} a_d q_1^d q_2 q_3 + 2\sum_{d\neq 1}\frac{b_d}{d-1} q_1^{d+1} q_2 q_3.
\end{aligned}
$$

Here as always, we dropped the trivial terms resulted from either the triviality of the invariants or the integrals being zero and in the penultimate equality, we replaced the basis elements by expressions involving quantum products.

We introduce power series

$$
\begin{aligned}
&\phi_0(q) = \sum_{d} a_d q^d, \qquad \phi_1(q) = \sum_{d\neq 2}\frac{a_d}{d-2} q^d,\\
&\psi_0(q) = \sum_{d\neq 1} b_d q^d, \quad \psi_1(q) = \sum_{d\neq 1}\frac{b_d}{d-1} q^d, \quad \psi_2(q) = \sum_{d\neq 1}\frac{b_d}{(d-1)^2} q^d.
\end{aligned}
$$

Then we have the expression

$$
\begin{aligned}
\beta_7 * \beta_{10} =&\beta_7\beta_{10} - \frac{1}{2}\psi_0(q_1)q_2\beta_{10} * \beta_{11} - \psi_1(q_1)q_2\beta_{11} * \beta_{11} + \psi_1(q_1)q_1q_2^2\beta_{11}\\
&- \psi_1(q_1)q_2\beta_9 + 2\phi_0(q_1)q_2q_3 + 2\psi_1(q_1)q_1q_2q_3.
\end{aligned}
$$

Similarly, ignoring vanishing terms, we have

$$\begin{aligned}
\beta_7 * \beta_{11} =& \beta_7\beta_{11} - \sum_d < \beta_7, \beta_4 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(-\frac{1}{2}\beta_5) \\
& - \sum_d < \beta_7, \beta_5 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
& - \sum_d < \beta_7, \beta_6 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2 \cdot \frac{1}{2}\beta_7 \\
& - \sum_d < \beta_7, \beta_7 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
& - \sum_d < \beta_7, \beta_8 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(\frac{1}{2}\beta_7 + \beta_8) \\
=& \beta_7\beta_{11} - \sum_{d\neq 1} \frac{b_d}{d-1} q_1^d q_2(-\frac{1}{2}\beta_5) + \sum_{d\neq 1} \frac{b_d}{d-1} q_1^d q_2(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
& + 2q_1q_2 \cdot \frac{1}{2}\beta_7 - (2q_1q_2 + \sum_{d\neq 1} \frac{b_d}{(d-1)^2} q_1^d q_2)(\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
& + \sum_{d\neq 1} \frac{b_d}{(d-1)^2} q_1^d q_2(\frac{1}{2}\beta_7 + \beta_8) \\
=& \beta_7\beta_{11} - \frac{1}{2}\psi_1(q_1)q_2\beta_4 - (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_6 + (-q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_8 \\
=& \beta_7\beta_{11} - \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} - (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11} * \beta_{11} \\
& - 2q_1q_2\beta_{11} * \beta_{12} + (-q_1^2q_2^2 + \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} \\
& + (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_9 + 2q_1^2q_2q_3 + \psi_2(q_1)q_1q_2q_3;
\end{aligned}$$

$$\begin{aligned}
\beta_7 * \beta_{12} =& \beta_7\beta_{12} + \sum_d < \beta_7, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2 q_3 \\
& + \sum_d < \beta_7, \beta_4 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(-\frac{1}{2}\beta_5) \\
& + \sum_d < \beta_7, \beta_5 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
& + \sum_d < \beta_7, \beta_6 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2 \cdot \frac{1}{2}\beta_7 \\
& + \sum_d < \beta_7, \beta_7 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
& + \sum_d < \beta_7, \beta_8 >_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2(\frac{1}{2}\beta_7 + \beta_8)
\end{aligned}$$

$$
\begin{aligned}
=&\beta_7\beta_{12} + 2q_1^2q_2q_3 - \sum_{d\neq 2}\frac{a_d}{d-2}q_1^dq_2q_3 + \sum_{d\neq 1}\frac{b_d}{d-1}q_1^dq_2(-\frac{1}{2}\beta_5)\\
&- \sum_{d\neq 1}\frac{b_d}{d-1}q_1^dq_2(-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) - 2q_1q_2\cdot\frac{1}{2}\beta_7\\
&+ (2q_1q_2 + \sum_{d\neq 1}\frac{b_d}{(d-1)^2}q_1^dq_2)(\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8)\\
&- \sum_{d\neq 1}\frac{b_d}{(d-1)^2}q_1^dq_2(\frac{1}{2}\beta_7 + \beta_8)\\
=&\beta_7\beta_{12} + 2q_1^2q_2q_3 - \phi_1(q_1)q_2q_3 + \frac{1}{2}\psi_1(q_1)q_2\beta_4 + (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_6\\
&+ (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_8\\
=&\beta_7\beta_{12} + \frac{1}{4}\psi_1(q_1)q_2\beta_{10}*\beta_{11} + (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11}*\beta_{11}\\
&+ 2q_1q_2\beta_{11}*\beta_{12} + (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} + (-q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_9\\
&- \psi_2(q_1)q_1q_2q_3 - \phi_1(q_1)q_2q_3;
\end{aligned}
$$

$$
\begin{aligned}
\beta_6*\beta_{10} =&\beta_6\beta_{10} - 2\sum_d(d-2) < \beta_6, pt >_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^dq_2q_3\\
&- 2\sum_d(d-1) < \beta_6, \beta_1 >_{d\beta_1+(\beta_3-\beta_1)}\ q_1^dq_3(-\frac{1}{2}\beta_{10})\\
&- 2\sum_d(d-1) < \beta_6, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)}\ q_1^dq_3\beta_{12}\\
&- 2\sum_d(d-1) < \beta_6, \beta_3 >_{d\beta_1+(\beta_3-\beta_1)}\ q_1^dq_3(\beta_{11}+\beta_{12})\\
&- 2\sum_d(d-1) < \beta_6, \beta_6 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^dq_2\cdot\frac{1}{2}\beta_7\\
&- 2\sum_d(d-1) < \beta_6, \beta_7 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^dq_2(\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8)\\
&- 2\sum_d(d-1) < \beta_6, \beta_8 >_{d\beta_1+(\beta_2-\beta_1)}\ q_1^dq_2(\frac{1}{2}\beta_7 + \beta_8)\\
=&\beta_6\beta_{10} + \sum_d(d-1)g_dq_1^dq_3\beta_{10} - 2\sum_d g_dq_1^dq_3\beta_{12}.
\end{aligned}
$$

If we introduce

$$\zeta_{-1}(q) = \sum_d (d-1) g_d q^d, \quad \zeta_0(q) = \sum_d g_d q^d, \quad \zeta_1(q) = \sum_{d\neq 1} \frac{g_d}{d-1} q^d,$$

then we have

$$\beta_6 * \beta_{10} = \beta_6\beta_{10} + \zeta_{-1}(q_1) q_3 \beta_{10} - 2\zeta_0(q_1) q_3 \beta_{12}.$$

Continuing the computations, we get

$$\begin{aligned}
\beta_6 * \beta_{11} =& \beta_6\beta_{11} + \sum_d < \beta_6, \beta_1 >_{d\beta_1 + (\beta_3 - \beta_1)} \; q_1^d q_3 (-\frac{1}{2}\beta_{10}) \\
&+ \sum_d < \beta_6, \beta_2 >_{d\beta_1 + (\beta_3 - \beta_1)} \; q_1^d q_3 \beta_{12} \\
&+ \sum_d < \beta_6, \beta_3 >_{d\beta_1 + (\beta_3 - \beta_1)} \; q_1^d q_3 (\beta_{11} + \beta_{12}) \\
&- \sum_d < \beta_6, \beta_6 >_{d\beta_1 + (\beta_2 - \beta_1)} \; q_1^d q_2 \cdot \frac{1}{2}\beta_7 \\
&- \sum_d < \beta_6, \beta_7 >_{d\beta_1 + (\beta_2 - \beta_1)} \; q_1^d q_2 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
&- \sum_d < \beta_6, \beta_8 >_{d\beta_1 + (\beta_2 - \beta_1)} \; q_1^d q_2 (\frac{1}{2}\beta_7 + \beta_8) \\
=& \beta_6\beta_{11} - \frac{1}{2}\sum_d g_d q_1^d q_3 \beta_{10} + (q_1 q_3 + \sum_{d\neq 1} \frac{g_d}{d-1} q_1^d q_3)\beta_{12} + 2q_1q_3(\beta_{11} + \beta_{12}) \\
&- \frac{1}{2} q_1 q_2 \beta_7 + q_1 q_2 (\beta_6 + \beta_7 + \beta_8) - q_1 q_2 (\frac{1}{2}\beta_7 + \beta_8) \\
=& \beta_6\beta_{11} - \frac{1}{2}\zeta_0(q_1) q_3 \beta_{10} + (3q_1q_3 + \zeta_1(q_1)q_3)\beta_{12} + 2q_1q_3\beta_{11} + q_1q_2\beta_6 \\
=& \beta_6\beta_{11} + q_1q_2\beta_{11} * \beta_{11} + q_1q_2\beta_{11} * \beta_{12} - \frac{1}{2}\zeta_0(q_1)q_3\beta_{10} \\
&+ (3q_1q_3 + \zeta_1(q_1)q_3)\beta_{12} + 2q_1q_3\beta_{11} - 2q_1^2 q_2 q_3;
\end{aligned}$$

$$
\begin{aligned}
\beta_8 * \beta_{10} =& \beta_8\beta_{10} - 2\sum_d (d-1) < \beta_8, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)} \, q_1^d q_3 \beta_{12} \\
& - 2\sum_d (d-1) < \beta_8, \beta_4 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (-\frac{1}{2}\beta_5) \\
& - 2\sum_d (d-1) < \beta_8, \beta_5 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
& - 2\sum_d (d-1) < \beta_8, \beta_6 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 \cdot \frac{1}{2}\beta_7 \\
& - 2\sum_d (d-1) < \beta_8, \beta_7 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
& - 2\sum_d (d-1) < \beta_8, \beta_8 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (\frac{1}{2}\beta_7 + \beta_8) \\
=& \beta_8\beta_{10} + 2\sum_{d\neq 1} b_d q_1^d q_2 (-\frac{1}{2}\beta_5) - 2\sum_{d\neq 1} b_d q_1^d q_2 (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
& + 2\sum_{d\neq 1} \frac{b_d}{d-1} q_1^d q_2 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) - 2\sum_{d\neq 1} \frac{b_d}{d-1} q_1^d q_2 (\frac{1}{2}\beta_7 + \beta_8) \\
=& \beta_8\beta_{10} + \psi_0(q_1) q_2 \beta_4 + \psi_1(q_1) q_2 \beta_6 - \psi_1(q_1) q_2 \beta_8 \\
=& \beta_8\beta_{10} + \frac{1}{2}\psi_0(q_1) q_2 \beta_{10} * \beta_{11} + \psi_1(q_1) q_2 \beta_{11} * \beta_{11} \\
& - \psi_1(q_1) q_1 q_2^2 \beta_{11} + \psi_1(q_1) q_2 \beta_9 - 2\psi_1(q_1) q_1 q_2 q_3;
\end{aligned}
$$

$$
\begin{aligned}
\beta_8 * \beta_{11} =& \beta_8\beta_{11} + \sum_d < \beta_8, \beta_2 >_{d\beta_1+(\beta_3-\beta_1)} \, q_1^d q_3 \beta_{12} \\
& - \sum_d < \beta_8, \beta_4 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (-\frac{1}{2}\beta_5) \\
& - \sum_d < \beta_8, \beta_5 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5) \\
& - \sum_d < \beta_8, \beta_6 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 \cdot \frac{1}{2}\beta_7 \\
& - \sum_d < \beta_8, \beta_7 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8) \\
& - \sum_d < \beta_8, \beta_8 >_{d\beta_1+(\beta_2-\beta_1)} \, q_1^d q_2 (\frac{1}{2}\beta_7 + \beta_8)
\end{aligned}
$$

$$
\begin{aligned}
=&\beta_8\beta_{11}+q_1q_3\beta_{12}+\sum_{d\neq 1}\frac{b_d}{d-1}q_1^dq_2(-\frac{1}{2}\beta_5)-\sum_{d\neq 1}\frac{b_d}{d-1}q_1^dq_2(-\frac{1}{2}\beta_4-\frac{1}{2}\beta_5)\\
&-q_1q_2\cdot\frac{1}{2}\beta_7+\sum_{d\neq 1}\frac{b_d}{(d-1)^2}q_1^dq_2(\frac{1}{2}\beta_6+\frac{1}{2}\beta_7+\frac{1}{2}\beta_8)\\
&+(q_1q_2-\sum_{d\neq 1}\frac{b_d}{(d-1)^2}q_1^dq_2)(\frac{1}{2}\beta_7+\beta_8)\\
=&\beta_8\beta_{11}+q_1q_3\beta_{12}+\frac{1}{2}\psi_1(q_1)q_2\beta_4+\frac{1}{2}\psi_2(q_1)q_2\beta_6+(q_1q_2-\frac{1}{2}\psi_2(q_1)q_2)\beta_8\\
=&\beta_8\beta_{11}+\frac{1}{4}\psi_1(q_1)q_2\beta_{10}*\beta_{11}+\frac{1}{2}\psi_2(q_1)q_2\beta_{11}*\beta_{11}+q_1q_2\beta_{11}*\beta_{12}\\
&+(q_1^2q_2^2-\frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11}+q_1q_3\beta_{12}+(\frac{1}{2}\psi_2(q_1)q_2-q_1q_2)\beta_9-\psi_2(q_1)q_1q_2q_3.
\end{aligned}
$$

By definition,

$$
\begin{aligned}
\beta_3*\beta_{11}=&\beta_3\beta_{11}+\sum_d<\beta_3,\beta_{11},pt>_{d\beta_1+4(\beta_2-\beta_1)}\ q_1^dq_2^4\\
&+\sum_d<\beta_3,\beta_{11},pt>_{d\beta_1+2(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^dq_2^2q_3\\
&+\sum_d<\beta_3,\beta_{11},pt>_{d\beta_1+2(\beta_3-\beta_1)}\ q_1^dq_3^2\\
&+\sum_d<\beta_3,\beta_{11},\beta_1>_{d\beta_1+3(\beta_2-\beta_1)}\ q_1^dq_2^3(-\frac{1}{2}\beta_{10})\\
&+\sum_d<\beta_3,\beta_{11},\beta_2>_{d\beta_1+3(\beta_2-\beta_1)}\ q_1^dq_2^3\beta_{12}\\
&+\sum_d<\beta_3,\beta_{11},\beta_3>_{d\beta_1+3(\beta_2-\beta_1)}\ q_1^dq_2^3(\beta_{11}+\beta_{12})\\
&+\sum_d<\beta_3,\beta_{11},\beta_1>_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^dq_2q_3(-\frac{1}{2}\beta_{10})\\
&+\sum_d<\beta_3,\beta_{11},\beta_2>_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^dq_2q_3\beta_{12}\\
&+\sum_d<\beta_3,\beta_{11},\beta_3>_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)}\ q_1^dq_2q_3(\beta_{11}+\beta_{12})
\end{aligned}
$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_4 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 (-\frac{1}{2}\beta_5)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_5 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_6 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 (\frac{1}{2}\beta_7)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_7 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_8 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 (\frac{1}{2}\beta_7 + \beta_8)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_9 >_{d\beta_1+2(\beta_2-\beta_1)} q_1^d q_2^2 \beta_9$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_4 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 (-\frac{1}{2}\beta_5)$$

$$+\sum_{d_1} < \beta_3, \beta_{11}, \beta_5 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 (-\frac{1}{2}\beta_4 - \frac{1}{2}\beta_5)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_6 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 (\frac{1}{2}\beta_7)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_7 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 (\frac{1}{2}\beta_6 + \frac{1}{2}\beta_7 + \frac{1}{2}\beta_8)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_8 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 (\frac{1}{2}\beta_7 + \beta_8)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_9 >_{d\beta_1+(\beta_3-\beta_1)} q_1^d q_3 \beta_9$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_{10} >_{d\beta_1+(\beta_2-\beta_1)} q_1^d q_2 (-\frac{1}{2}\beta_1)$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_{11} >_{d\beta_1+(\beta_2-\beta_1)} q_1^d q_2 \beta_3$$

$$+\sum_{d} < \beta_3, \beta_{11}, \beta_{12} >_{d\beta_1+(\beta_2-\beta_1)} q_1^d q_2 (\beta_2 + \beta_3)$$

$$=\beta_3\beta_{11} + q_1q_3\beta_7 + q_1q_2\beta_3 - q_1q_2(\beta_2+\beta_3) = \beta_3\beta_{11} + q_1q_3\beta_7 - q_1q_2\beta_2$$

$$=\beta_3\beta_{11} + q_1q_3\beta_{12} * \beta_{12} - q_1q_2\beta_9 * \beta_{11} - q_1^2q_2q_3\beta_{11} + q_1^2q_2q_3\beta_{12}.$$

Similarly, with β_{11} replaced by β_{12},

$$\begin{aligned}
\beta_3 * \beta_{12} =& \beta_3\beta_{12} + <\beta_3, \beta_1>_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2 q_3 (-\frac{1}{2}\beta_{10}) \\
&+ <\beta_3, \beta_2>_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2 q_3 \beta_{12} \\
&+ <\beta_3, \beta_3>_{d\beta_1+(\beta_2-\beta_1)+(\beta_3-\beta_1)} \; q_1^d q_2 q_3 (\beta_{11} + \beta_{12}) \\
&+ <\beta_3, \beta_{10}>_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2 (-\frac{1}{2}\beta_1) \\
&+ <\beta_3, \beta_{11}>_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2 \beta_3 \\
&+ <\beta_3, \beta_{12}>_{d\beta_1+(\beta_2-\beta_1)} \; q_1^d q_2 (\beta_2 + \beta_3) \\
=& \beta_3\beta_{12} + \frac{1}{2}\sum_{d\neq 2} \frac{c_d}{d-2} q_1^d q_2 q_3 \beta_{10} + q_1^2 q_2 q_3 \beta_{12} \\
&+ (2q_1^2 q_2 q_3 + \sum_{d\neq 2} \frac{c_d}{(d-2)^2} q_1^d q_2 q_3)(\beta_{11} + \beta_{12}) + q_1 q_2 \beta_2.
\end{aligned}$$

We define

$$\xi_0(q) = \sum_d c_d q^d, \quad \xi_1(q) = \sum_{d\neq 2} \frac{c_d}{d-2} q^d, \quad \xi_2(q) = \sum_{d\neq 2} \frac{c_d}{(d-2)^2} q^d,$$

then

$$\begin{aligned}
\beta_3 * \beta_{12} =& \beta_3\beta_{12} + \frac{1}{2}\xi_1(q_1) q_2 q_3 \beta_{10} + q_1^2 q_2 q_3 \beta_{12} \\
&+ (2q_1^2 q_2 q_3 + \xi_2(q_1) q_2 q_3)(\beta_{11} + \beta_{12}) + q_1 q_2 \beta_2 \\
=& \beta_3\beta_{12} + q_1 q_2 \beta_9 * \beta_{11} + \frac{1}{2}\xi_1(q_1) q_2 q_3 \beta_{10} \\
&+ (2q_1^2 q_2 q_3 + \xi_2(q_1) q_2 q_3)\beta_{11} + (2q_1^2 q_2 q_3 + \xi_2(q_1) q_2 q_3)\beta_{12}.
\end{aligned}$$

5.4 Presentation of Quantum Product

Suppose for a smooth variety X, $H_2(X, \mathbb{Z})$ is freely generated by effective classes $q_1, \cdots, q_k, q_{k+1}, \cdots, q_{k+l}$, where $degq_i = 0$, for $i = 1, \cdots, k$, and $degq_{k+j} \geq 1$, for $j = 1, \cdots, l$, where here $degq_i$ means $\int_{q_i} c_1(X)$. Then quantum product of X is closed when restricted to the subring $H^*(X, \mathbb{Q}) \otimes \mathbb{Q}[q_{k+1}, \cdots, q_{k+l}][[q_1, \cdots, q_k]]$ in the quantum cohomology ring $H^*(X, \mathbb{Q}) \otimes \mathbb{Q}[[q_1, \cdots, q_{k+l}]]$ and it is obvious that to determine the ring structure of the latter, it is sufficient to determine the structure of the former. But first we give a name to it.

Definition 5.4.1. *Let $QH^*(X) = H^*(X, \mathbb{Q}) \otimes \mathbb{Q}[q_{k+1}, \cdots, q_{k+l}][[q_1, \cdots, q_k]]$. We still call this ring the (small) quantum cohomology ring.*

Suppose

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[Z_1, \cdots, Z_n]/(f_1, \cdots, f_s),$$

is a presentation of the cohomology ring of X, where $Z_1, \cdots, Z_n$ are liftings of some basis elements with respective degrees and form a set of generators in $H^*(X, \mathbb{Q})$ and $f_1, \cdots, f_s$ are homogeneous generators for the ideal of relations.

Let

$$\mathbb{Q}(q) = \mathbb{Q}[q_{k+1}, \cdots, q_{k+l}][[q_1, \cdots, q_k]],$$
$$\mathbb{Q}(q, Z) = \mathbb{Q}[q_{k+1}, \cdots, q_{k+l}, Z_1, \cdots, Z_n][[q_1, \cdots, q_k]].$$

Then we have the following proposition, which is a modification of a result from [16].

Proposition 5.4.2. *Let $f'_1, \cdots, f'_s$ be homogeneous elements in $\mathbb{Q}(q, Z)$ such that*

$(i) f'_i(0, \cdots, 0, Z_1, \cdots, Z_n) = f_i(Z_1, \cdots, Z_n)$ in $\mathbb{Q}(q, Z)$,

$(ii) f'_i(q_1, \cdots, q_{k+l}, Z_1, \cdots, Z_n) = 0$ in $QH^*(X)$.

Then the canonical map

$$\phi : \mathbb{Q}(q, Z)/(f'_1, \cdots, f'_s) \longrightarrow QH^*(X)$$

is surjective. If, in addition, $\mathbb{Q}(q, Z)/(f'_1, \cdots, f'_s)$ is finitely generated over $\mathbb{Q}(q)$, then this map is an isomorphism.

Before we prove the proposition, we first state

Lemma 5.4.3. *Assume that M is a non-negatively graded module over a graded ring R, I is an ideal in R generated by elements of positive degrees. Then we have* (*i*) *If* $IM = M$*, then* $M = 0$.

(*ii*) *If* $N \subset M$ *is a graded submodule and* $M = N + IM$*, then* $M = N$.

Proof. (*i*) Suppose otherwise $M \neq 0$. Then we can take a nontrivial homogeneous element, say a, in M of the lowest degree. From $IM = M$, a can be written as a linear combination of elements with coefficients in I. But these coefficients have positive degrees by the assumption on I so the elements of M appearing in this combination must have lower degrees. This constitutes a contradiction, thus finishing the proof.

(*ii*) follows from (*i*) applied to M/N. □

Now we begin the proof of the proposition.

Proof. First we note if $\psi : M \to N$ is a graded homomorphism between two finitely generated non-negatively graded $\mathbb{Q}(q, Z)$-modules such that the induced map

$$\frac{M/(q_1, \cdots, q_k)}{(q_{k+1}, \cdots, q_{k+l})} \to \frac{N/(q_1, \cdots, q_k)}{(q_{k+1}, \cdots, q_{k+l})}$$

is surjective, then

$$M/(q_1, \cdots, q_k) \to N/(q_1, \cdots, q_k),$$

is also surjective by (*ii*) of the above lemma since $(q_{k+1}, \cdots, q_{k+l})$ is an ideal in $\mathbb{Q}(q, Z)$ generated by elements of positive degrees . By Nakayama's lemma, $\psi : M \to N$ is surjective since $(q_1, \cdots, q_k)$ is contained in the Jacobson radical of $\mathbb{Q}(q, Z)$. This observation combined with hypothesis (*i*) gives rise to the surjectivity of the map ϕ in the proposition.

Now assume that $\mathbb{Q}(q, Z)/(f'_1, \cdots, f'_s)$ is finitely generated over $\mathbb{Q}(q)$. Suppose $W_i, i = 1, \cdots, m$ are lifted elements to $\mathbb{Q}(q, Z)/(f'_1, \cdots, f'_s)$ from basis elements of $H^*(X, \mathbb{Q})$, where $m = dimH^*(X, \mathbb{Q})$. Then the analogous argument as above shows that they generate $\mathbb{Q}(q, Z)/(f'_1, \cdots, f'_s)$ as a $\mathbb{Q}(q)$-module. In other words, this module is generated by at most m elements. On the other hand, it is obvious that $QH^*(X)$ is of rank m as a free $\mathbb{Q}(q)$-module. Then that ϕ is an isomorphism results from the following lemma. □

Lemma 5.4.4. *For a ring R, if M is generated by r elements as an R-module and there is a surjective map $M \to R^r$, then M is free of rank r.*

Proof. We use f to denote the map $M \to R^r$. That M is generated by r elements as an R-module means there is a surjective map $g : R^r \to M$. Then the composition $fg : R^r \to R^r$ is also surjective. Now theorem 2.4 in [14] tells us that this composition must be injective, which forces g to be an injection hence an isomorphism, which in turn implies that f is an isomorphism. $\square$

Now we apply this result to the computation of quantum cohomology ring of $F^{[2]}$. First we note $degq_1 = 0, degq_2 = 1, degq_3 = 2$, so here $\mathbb{Q}(q) = \mathbb{Q}[q_2, q_3][[q_1]]$. We have derived a set of generators of the ideal of relations for $H^*(F^{[2]}, \mathbb{Q})$, which are listed at the beginning of §5.1. What we need to do in this section is to find the set f_i' satisfying the conditions in the proposition using the results we have obtained so far, thus completely determining the ring structure of $QH^*(F^{[2]})$.

We start with the relation $P1 : \beta_{10}^2 - 2\beta_{10}\beta_{11} - 3\beta_{10}\beta_{12} + 2\beta_{12}^2 + 4\beta_{11}\beta_{12} = 0$. In the expression (5.1)

$$\beta_{10} * \beta_{10} = \beta_{10}^2 + 8\sum_{d\neq 0} dq_1^d\beta_4 + 12\sum_{d\neq 0} dq_1^d\beta_5,$$

we plug in

$$\beta_4 = \frac{1}{2}\beta_{10} * \beta_{11}, \quad \beta_5 = \frac{1}{2}\beta_{10} * \beta_{12},$$

from (5.2) to get

$$\beta_{10}^2 = \beta_{10} * \beta_{10} - 4\sum_{d_1\neq 0} d_1q_1^{d_1}\beta_{10} * \beta_{11} - 6\sum_{d_1\neq 0} d_1q_1^{d_1}\beta_{10} * \beta_{12}.$$

Also,

$$\begin{aligned} \beta_{10}\beta_{11} &= \beta_{10} * \beta_{11}, & \beta_{10}\beta_{12} &= \beta_{10} * \beta_{12}, \\ \beta_{12}^2 &= \beta_{12} * \beta_{12} - q_1q_2\beta_{11}, & \beta_{11}\beta_{12} &= \beta_{11} * \beta_{12} + q_1q_2\beta_{11}, \end{aligned}$$

Substituting all these into the equation $P1$, we obtain one relation

$$\begin{aligned}R1: \ & \beta_{10} * \beta_{10} - 2\beta_{10} * \beta_{11} - 3\beta_{10} * \beta_{12} + 2\beta_{12} * \beta_{12} + 4\beta_{11} * \beta_{12} \\ & + 2q_1q_2\beta_{11} - 4\sum_{d_1 \neq 0} d_1 q_1^{d_1}\beta_{10} * \beta_{11} - 6\sum_{d_1 \neq 0} d_1 q_1^{d_1}\beta_{10} * \beta_{12} = 0.\end{aligned}$$

Then working with $P2: \beta_{10}\beta_{12}^2 = 0$, we see

$$\begin{aligned}&\beta_{10} * \beta_{12} * \beta_{12} = \beta_{10} * (\beta_{12} * \beta_{12}) \\ =&\beta_{10} * (\beta_{12}^2 + q_1q_2\beta_{11}) = \beta_7 * \beta_{10} + q_1q_2\beta_{10} * \beta_{11} \\ =&\beta_7\beta_{10} - \frac{1}{2}\psi_0(q_1)q_2\beta_{10} * \beta_{11} - \psi_1(q_1)q_2\beta_{11} * \beta_{11} + \psi_1(q_1)q_1q_2^2\beta_{11} \\ &- \psi_1(q_1)q_2\beta_9 + 2\phi_0(q_1)q_2q_3 + 2\psi_1(q_1)q_1q_2q_3 + q_1q_2\beta_{10} * \beta_{11} \\ =&\beta_{10}\beta_{12}^2 + (q_1q_2 - \frac{1}{2}\psi_0(q_1)q_2)\beta_{10} * \beta_{11} - \psi_1(q_1)q_2\beta_{11} * \beta_{11} \\ &+ \psi_1(q_1)q_1q_2^2\beta_{11} - \psi_1(q_1)q_2\beta_9 + 2\phi_0(q_1)q_2q_3 + 2\psi_1(q_1)q_1q_2q_3,\end{aligned}$$

then we get the relation

$$\begin{aligned}R2: \beta_{10}*&\beta_{12} * \beta_{12} - (q_1q_2 - \frac{1}{2}\psi_0(q_1)q_2)\beta_{10} * \beta_{11} + \psi_1(q_1)q_2\beta_{11} * \beta_{11} \\ &- \psi_1(q_1)q_1q_2^2\beta_{11} + \psi_1(q_1)q_2\beta_9 - 2\phi_0(q_1)q_2q_3 - 2\psi_1(q_1)q_1q_2q_3 = 0.\end{aligned}$$

For $P3: \beta_{12}^3 = 0$,

$$\begin{aligned}&\beta_{12} * \beta_{12} * \beta_{12} = \beta_{12} * (\beta_{12}^2 + q_1q_2\beta_{11}) \\ =&\beta_7 * \beta_{12} + q_1q_2\beta_{11} * \beta_{12} \\ =&\beta_7\beta_{12} + \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} + (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11} * \beta_{11} + 2q_1q_2\beta_{11} * \beta_{12} \\ &+ (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} + (-q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_9 - \psi_2(q_1)q_1q_2q_3 \\ &- \phi_1(q_1)q_2q_3 + q_1q_2\beta_{11} * \beta_{12} \\ =&\beta_{12}^3 + \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} + (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11} * \beta_{11} + 3q_1q_2\beta_{11} * \beta_{12} \\ &+ (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} + (-q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_9 - \psi_2(q_1)q_1q_2q_3 - \phi_1(q_1)q_2q_3,\end{aligned}$$

then we get the relation

$$
\begin{aligned}
R3: \beta_{12}*\beta_{12}*\beta_{12} &- \frac{1}{4}\psi_1(q_1)q_2\beta_{10}*\beta_{11} - (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11}*\beta_{11} \\
&- 3q_1q_2\beta_{11}*\beta_{12} - (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} \\
&+ (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_9 + \psi_2(q_1)q_1q_2q_3 + \phi_1(q_1)q_2q_3 = 0.
\end{aligned}
$$

For $P4: \beta_{11}\beta_{12}^2 - 2\beta_9\beta_{12} = 0$,

$$
\begin{aligned}
&\beta_{11}*\beta_{12}*\beta_{12} - 2\beta_9*\beta_{12} \\
=&\beta_{11}*(\beta_{12}^2 + q_1q_2\beta_{11}) - 2(\beta_9\beta_{12} + 2q_1^2q_2q_3) \\
=&\beta_7*\beta_{11} + q_1q_2\beta_{11}*\beta_{11} - 2\beta_9\beta_{12} - 4q_1^2q_2q_3 \\
=&\beta_7\beta_{11} - \frac{1}{4}\psi_1(q_1)q_2\beta_{10}*\beta_{11} - (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11}*\beta_{11} - 2q_1q_2\beta_{11}*\beta_{12} \\
&+ (-q_1^2q_2^2 + \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} + (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_9 + 2q_1^2q_2q_3 \\
&+ \psi_2(q_1)q_1q_2q_3 + q_1q_2\beta_{11}*\beta_{11} - 2\beta_9\beta_{12} - 4q_1^2q_2q_3 \\
=&\beta_{11}\beta_{12}^2 - 2\beta_9\beta_{12} - \frac{1}{4}\psi_1(q_1)q_2\beta_{10}*\beta_{11} - \frac{1}{2}\psi_2(q_1)q_2\beta_{11}*\beta_{11} - 2q_1q_2\beta_{11}*\beta_{12} \\
&+ (-q_1^2q_2^2 + \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} + (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_9 - 2q_1^2q_2q_3 + \psi_2(q_1)q_1q_2q_3,
\end{aligned}
$$

so we get the relation

$$
\begin{aligned}
R4: \beta_{11}*\beta_{12}*\beta_{12} &- 2\beta_9*\beta_{12} + \frac{1}{4}\psi_1(q_1)q_2\beta_{10}*\beta_{11} + \frac{1}{2}\psi_2(q_1)q_2\beta_{11}*\beta_{11} \\
&+ 2q_1q_2\beta_{11}*\beta_{12} + (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} \\
&+ (-q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_9 + 2q_1^2q_2q_3 - \psi_2(q_1)q_1q_2q_3 = 0.
\end{aligned}
$$

For $P5 : \beta_{10}\beta_{11}^2 + 2\beta_9\beta_{10} = 0$,

$$
\begin{aligned}
&\beta_{10} * \beta_{11} * \beta_{11} + 2\beta_9 * \beta_{10} \\
=&\beta_{10} * (\beta_{11}^2 + q_1q_2\beta_{11} + 2q_1q_3) + 2\beta_9\beta_{10} \\
=&\beta_{10} * (\beta_6 - \beta_8 - \beta_9 + q_1q_2\beta_{11} + 2q_1q_3) + 2\beta_9\beta_{10} \\
=&\beta_6 * \beta_{10} - \beta_8 * \beta_{10} - \beta_9 * \beta_{10} + q_1q_2\beta_{10} * \beta_{11} + 2q_1q_3\beta_{10} + 2\beta_9\beta_{10} \\
=&\beta_6\beta_{10} + \zeta_{-1}(q_1)q_3\beta_{10} - 2\zeta_0(q_1)q_3\beta_{12} - \beta_8\beta_{10} - \frac{1}{2}\psi_0(q_1)q_2\beta_{10} * \beta_{11} \\
&- \psi_1(q_1)q_2\beta_{11} * \beta_{11} + \psi_1(q_1)q_1q_2^2\beta_{11} - \psi_1(q_1)q_2\beta_9 + 2\psi_1(q_1)q_1q_2q_3 \\
&- \beta_9\beta_{10} + q_1q_2\beta_{10} * \beta_{11} + 2q_1q_3\beta_{10} + 2\beta_9\beta_{10} \\
=&\beta_{10}\beta_{11}^2 + 2\beta_9\beta_{10} + \zeta_{-1}(q_1)q_3\beta_{10} - 2\zeta_0(q_1)q_3\beta_{12} \\
&- \frac{1}{2}\psi_0(q_1)q_2\beta_{10} * \beta_{11} - \psi_1(q_1)q_2\beta_{11} * \beta_{11} + \psi_1(q_1)q_1q_2^2\beta_{11} \\
&- \psi_1(q_1)q_2\beta_9 + 2\psi_1(q_1)q_1q_2q_3 + q_1q_2\beta_{10} * \beta_{11} + 2q_1q_3\beta_{10},
\end{aligned}
$$

so we get the relation

$$
\begin{aligned}
R5 : \beta_{10}*\beta_{11} * \beta_{11} + 2\beta_9 * \beta_{10} &- (2q_1q_3 + \zeta_{-1}(q_1)q_3)\beta_{10} + 2\zeta_0(q_1)q_3\beta_{12} \\
&+ (\frac{1}{2}\psi_0(q_1)q_2 - q_1q_2)\beta_{10} * \beta_{11} + \psi_1(q_1)q_2\beta_{11} * \beta_{11} \\
&- \psi_1(q_1)q_1q_2^2\beta_{11} + \psi_1(q_1)q_2\beta_9 - 2\psi_1(q_1)q_1q_2q_3 = 0.
\end{aligned}
$$

For $P6 : \beta_{10}\beta_{11}\beta_{12} - 2\beta_9\beta_{10} = 0,$

$$
\begin{aligned}
&\beta_{10} * \beta_{11} * \beta_{12} - 2\beta_9 * \beta_{10} \\
=&\beta_{10} * (\beta_{11}\beta_{12} - q_1q_2\beta_{11}) - 2\beta_9\beta_{10} \\
=&\beta_{10} * (\beta_8 + \beta_9) - q_1q_2\beta_{10} * \beta_{11} - 2\beta_9\beta_{10} \\
=&\beta_8 * \beta_{10} + \beta_9 * \beta_{10} - q_1q_2\beta_{10} * \beta_{11} - 2\beta_9\beta_{10} \\
=&\beta_8\beta_{10} + \frac{1}{2}\psi_0(q_1)q_2\beta_{10} * \beta_{11} + \psi_1(q_1)q_2\beta_{11} * \beta_{11} \\
&- \psi_1(q_1)q_1q_2^2\beta_{11} + \psi_1(q_1)q_2\beta_9 - 2\psi_1(q_1)q_1q_2q_3 \\
&+ \beta_9\beta_{10} - q_1q_2\beta_{10} * \beta_{11} - 2\beta_9\beta_{10} \\
=&\beta_{10}\beta_{11}\beta_{12} - 2\beta_9\beta_{10} + \frac{1}{2}\psi_0(q_1)q_2\beta_{10} * \beta_{11} + \psi_1(q_1)q_2\beta_{11} * \beta_{11} \\
&- \psi_1(q_1)q_1q_2^2\beta_{11} + \psi_1(q_1)q_2\beta_9 - 2\psi_1(q_1)q_1q_2q_3 - q_1q_2\beta_{10} * \beta_{11},
\end{aligned}
$$

so we get the relation

$$
\begin{aligned}
R6 : \beta_{10}*&\beta_{11} * \beta_{12} - 2\beta_9 * \beta_{10} + (q_1q_2 - \frac{1}{2}\psi_0(q_1)q_2)\beta_{10} * \beta_{11} \\
&- \psi_1(q_1)q_2\beta_{11} * \beta_{11} + \psi_1(q_1)q_1q_2^2\beta_{11} - \psi_1(q_1)q_2\beta_9 + 2\psi_1(q_1)q_1q_2q_3 = 0.
\end{aligned}
$$

For $P7 : \beta_{11}^2\beta_{12} - 2\beta_9\beta_{11} + \beta_9\beta_{12} = 0,$

$$
\begin{aligned}
&\beta_{11} * \beta_{11} * \beta_{12} - 2\beta_9 * \beta_{11} + \beta_9 * \beta_{12} \\
=&\beta_{11} * (\beta_{11}\beta_{12} - q_1q_2\beta_{11}) - 2(\beta_9\beta_{11} + q_1q_3\beta_{12}) + \beta_9\beta_{12} + 2q_1^2q_2q_3 \\
=&\beta_8 * \beta_{11} + \beta_9 * \beta_{11} - q_1q_2\beta_{11} * \beta_{11} - 2q_1q_3\beta_{12} + 2q_1^2q_2q_3 - 2\beta_9\beta_{11} + \beta_9\beta_{12} \\
=&\beta_8\beta_{11} + \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} + \frac{1}{2}\psi_2(q_1)q_2\beta_{11} * \beta_{11} + q_1q_2\beta_{11} * \beta_{12} \\
&+ (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} - q_1q_3\beta_{12} + (\frac{1}{2}\psi_2(q_1)q_2 - q_1q_2)\beta_9 \\
&- \psi_2(q_1)q_1q_2q_3 + \beta_9\beta_{11} - q_1q_2\beta_{11} * \beta_{11} + 2q_1^2q_2q_3 - 2\beta_9\beta_{11} + \beta_9\beta_{12}
\end{aligned}
$$

$$
\begin{aligned}
=&\beta_{11}^2\beta_{12} - 2\beta_9\beta_{11} + \beta_9\beta_{12} + \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} + (-q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11} * \beta_{11} \\
&+ q_1q_2\beta_{11} * \beta_{12} + (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} - q_1q_3\beta_{12} \\
&+ (\frac{1}{2}\psi_2(q_1)q_2 - q_1q_2)\beta_9 + 2q_1^2q_2q_3 - \psi_2(q_1)q_1q_2q_3,
\end{aligned}
$$

so we get the relation

$$
\begin{aligned}
R7 : \beta_{11}{*}\beta_{11} * \beta_{12} - 2\beta_9 * \beta_{11} + \beta_9 * \beta_{12} - \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} \\
+ (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_{11} * \beta_{11} - q_1q_2\beta_{11} * \beta_{12} - (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} \\
+ q_1q_3\beta_{12} + (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_9 + \psi_2(q_1)q_1q_2q_3 - 2q_1^2q_2q_3 = 0.
\end{aligned}
$$

For $P8 : \beta_{11}^3 + 3\beta_9\beta_{11} = 0$,

$$
\begin{aligned}
&\beta_{11} * \beta_{11} * \beta_{11} + 3\beta_9 * \beta_{11} \\
=&\beta_{11} * (\beta_{11}^2 + q_1q_2\beta_{11} + 2q_1q_3) + 3(\beta_9\beta_{11} + q_1q_3\beta_{12}) \\
=&\beta_6 * \beta_{11} - \beta_8 * \beta_{11} - \beta_9 * \beta_{11} + q_1q_2\beta_{11} * \beta_{11} \\
&+ 2q_1q_3\beta_{11} + 3\beta_9\beta_{11} + 3q_1q_3\beta_{12} \\
=&\beta_6\beta_{11} + q_1q_2\beta_{11} * \beta_{11} + q_1q_2\beta_{11} * \beta_{12} - \frac{1}{2}\zeta_0(q_1)q_3\beta_{10} \\
&+ (3q_1q_3 + \zeta_1(q_1)q_3)\beta_{12} + 2q_1q_3\beta_{11} - 2q_1^2q_2q_3 \\
&- [\beta_8\beta_{11} + \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} + \frac{1}{2}\psi_2(q_1)q_2\beta_{11} * \beta_{11} + q_1q_2\beta_{11} * \beta_{12} \\
&+ (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} + q_1q_3\beta_{12} + (\frac{1}{2}\psi_2(q_1)q_2 - q_1q_2)\beta_9 - \psi_2(q_1)q_1q_2q_3] \\
&- (\beta_9\beta_{11} + q_1q_3\beta_{12}) + q_1q_2\beta_{11} * \beta_{11} + 2q_1q_3\beta_{11} + 3\beta_9\beta_{11} + 3q_1q_3\beta_{12} \\
=&\beta_{11}^3 + 3\beta_9\beta_{11} + (2q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_{11} * \beta_{11} - \frac{1}{2}\zeta_0(q_1)q_3\beta_{10} \\
&+ (4q_1q_3 + \zeta_1(q_1)q_3)\beta_{12} - \frac{1}{4}\psi_1(q_1)q_2\beta_{10} * \beta_{11} + (4q_1q_3 - q_1^2q_2^2 + \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} \\
&- (\frac{1}{2}\psi_2(q_1)q_2 - q_1q_2)\beta_9 + \psi_2(q_1)q_1q_2q_3 - 2q_1^2q_2q_3,
\end{aligned}
$$

so we get the relation

$$
\begin{aligned}
R8: \beta_{11}*\beta_{11}*\beta_{11} + 3\beta_9*\beta_{11} + \frac{1}{4}\psi_1(q_1)q_2\beta_{10}*\beta_{11} + (\frac{1}{2}\psi_2(q_1)q_2 - 2q_1q_2)\beta_{11}*\beta_{11} \\
+ \frac{1}{2}\zeta_0(q_1)q_3\beta_{10} + (q_1^2q_2^2 - 4q_1q_3 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} - (4q_1q_3 + \zeta_1(q_1)q_3)\beta_{12} \\
+ (\frac{1}{2}\psi_2(q_1)q_2 - q_1q_2)\beta_9 + 2q_1^2q_2q_3 - \psi_2(q_1)q_1q_2q_3 = 0.
\end{aligned}
$$

For $P9: \beta_9\beta_{11}\beta_{12} - \beta_9^2 = 0$,

$$
\begin{aligned}
&\beta_9*\beta_{11}*\beta_{12} - \beta_9*\beta_9 = (\beta_9*\beta_{12})*\beta_{11} - \beta_9*\beta_9 \\
=&\beta_3*\beta_{11} + 2q_1^2q_2q_3\beta_{11} - (\beta_9^2 + q_1^2q_2q_3\beta_{11} + q_1^2q_2q_3\beta_{12}) \\
=&\beta_3\beta_{11} + q_1q_3\beta_{12}*\beta_{12} - q_1q_2\beta_9*\beta_{11} - q_1^2q_2q_3\beta_{11} \\
&+ q_1^2q_2q_3\beta_{12} + 2q_1^2q_2q_3\beta_{11} - (\beta_9^2 + q_1^2q_2q_3\beta_{11} + q_1^2q_2q_3\beta_{12}) \\
=&\beta_9\beta_{11}\beta_{12} - \beta_9^2 + q_1q_3\beta_{12}*\beta_{12} - q_1q_2\beta_9*\beta_{11},
\end{aligned}
$$

so we get the relation

$$
R9: \beta_9*\beta_{11}*\beta_{12} - \beta_9*\beta_9 - q_1q_3\beta_{12}*\beta_{12} + q_1q_2\beta_9*\beta_{11} = 0.
$$

We use $f_1', f_2', \cdots, f_9'$ to denote the polynomials on the left-hand sides of the equations $R_1, R_2, \cdots$, R_9, respectively. Then by the above proposition we have the surjective map from the ring $\mathbb{Q}[\beta_9, \beta_{10}, \beta_{11}, \beta_{12}, q_2, q_3][[q_1]]/(f_1', f_2', \cdots, f_9')$ to the quantum cohomology ring. To prove this map is an isomorphism from the above proposition, we need

Lemma 5.4.5. *$\mathbb{Q}[\beta_9, \beta_{10}, \beta_{11}, \beta_{12}, q_2, q_3][[q_1]]/(f_1', f_2', \cdots, f_9')$ is finitely generated over $\mathbb{Q}[q_2, q_3][[q_1]]$.*

Proof. From $f_1', f_2', \cdots, f_8'$, we see that any power of $\beta_{10}, \beta_{11}, \beta_{12}$ of degree 3 in the quotient ring can be reduced to a linear combination over $\mathbb{Q}[q_2, q_3][[q_1]]$ of $1, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}$ and products among $\beta_9, \beta_{10}, \beta_{11}, \beta_{12}$ of the second degree excluding β_9^2 and β_{10}^2, in which the highest degree terms with β_9 are $\beta_9\beta_i$, $i = 10, 11, 12$. If continued, for any power of $\beta_{10}, \beta_{11}, \beta_{12}$ of degree 4, the highest degree terms with β_9 are $\beta_9\beta_i\beta_j$, $i, j = 10, 11, 12$. By iteration, any degree n monomial in $\beta_{10}, \beta_{11}, \beta_{12}$ in the quotient ring can be expressed as a linear combination of elements of the forms

$$(L_1)\quad \beta_9^k, \beta_9^k\beta_{10}, \beta_9^k\beta_{11}, \beta_9^k\beta_{12}, \beta_9^k\beta_{10}\beta_{11}, \beta_9^k\beta_{10}\beta_{12}, \beta_9^k\beta_{11}\beta_{12}, \beta_9^k\beta_{11}^2, \beta_9^k\beta_{12}^2,$$

where $k \geq 0$ and when $n \geq 3$, the degree of β_9 appearing in this combination is at most the integer part of $\frac{n-1}{2}$, or $[\frac{n-1}{2}]$. Here the word "degree" means that as polynomials. Obviously, any monomial in $\beta_9, \beta_{10}, \beta_{11}, \beta_{12}$ in the quotient ring can also be expressed by these elements.

On the other hand from f_9', we see

$$\beta_9^2 = \beta_9\beta_{11}\beta_{12} - q_1q_3\beta_{12}^2 + q_1q_2\beta_9\beta_{11},$$

so recursively,

$$\begin{aligned}
\beta_9^3 &= \beta_9^2\beta_{11}\beta_{12} - q_1q_3\beta_9\beta_{12}^2 + q_1q_2\beta_9^2\beta_{11} \\
&= \beta_9\beta_{11}^2\beta_{12}^2 - q_1q_3\beta_{11}\beta_{12}^3 + q_1q_2\beta_9\beta_{11}^2\beta_{12} - q_1q_3\beta_9\beta_{12}^2 + q_1q_2\beta_9^2\beta_{11}, \\
\beta_9^4 &= \beta_9^2\beta_{11}^2\beta_{12}^2 - q_1q_3\beta_9\beta_{11}\beta_{12}^3 + q_1q_2\beta_9^2\beta_{11}^2\beta_{12} - q_1q_3\beta_9^2\beta_{12}^2 + q_1q_2\beta_9^3\beta_{11} \\
&= \beta_9\beta_{11}^3\beta_{12}^3 - q_1q_3\beta_{11}^2\beta_{12}^4 + q_1q_2\beta_9\beta_{11}^3\beta_{12}^2 - q_1q_3\beta_9\beta_{11}\beta_{12}^3 \\
&\qquad + q_1q_2\beta_9^2\beta_{11}^2\beta_{12} - q_1q_3\beta_9^2\beta_{12}^2 + q_1q_2\beta_9^3\beta_{11}
\end{aligned}$$

From f_3', we know

$$\begin{aligned}
\beta_{12}^3 = &\frac{1}{4}\psi_1(q_1)q_2\beta_{10}\beta_{11} + (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_{11}^2 \\
&+ 3q_1q_2\beta_{11}\beta_{12} + (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_{11} \\
&- (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_9 - \psi_2(q_1)q_1q_2q_3 - \phi_1(q_1)q_2q_3.
\end{aligned}$$

Plugging this in the equation for β_9^4, we get

$$\begin{aligned}\beta_9^4 &= \frac{1}{4}\psi_1(q_1)q_2\beta_9\beta_{10}\beta_{11}^4 + (q_1q_2 + \frac{1}{2}\psi_2(q_1)q_2)\beta_9\beta_{11}^5 \\ &+ 3q_1q_2\beta_9\beta_{11}^4\beta_{12} + (q_1^2q_2^2 - \frac{1}{2}\psi_2(q_1)q_1q_2^2)\beta_9\beta_{11}^4 \\ &- (q_1q_2 - \frac{1}{2}\psi_2(q_1)q_2)\beta_9^2\beta_{11}^3 - (\psi_2(q_1)q_1q_2q_3 + \phi_1(q_1)q_2q_3)\beta_9\beta_{11}^3 \\ &- q_1q_3\beta_{11}^2\beta_{12}^4 + q_1q_2\beta_9\beta_{11}^3\beta_{12}^2 - q_1q_3\beta_9\beta_{11}\beta_{12}^3 + q_1q_2\beta_9^2\beta_{11}^2\beta_{12} \\ &\qquad - q_1q_3\beta_9^2\beta_{12}^2 + q_1q_2\beta_9^3\beta_{11}.\end{aligned}$$

All the monomials appearing in the right-hand side of this equation have nontrivial coefficients in q_2, q_3, which implies their degrees as in quantum cohomology ring are at least 1. When their parts of products of $\beta_{10}, \beta_{11}, \beta_{12}$ are reduced by $f_1', f_2', \cdots, f_8'$ described at the beginning of this proof, we realize β_9^4 as a linear combination over $\mathbb{Q}[q_2, q_3][[q_1]]$ of elements of the following forms

$$(L_2) \qquad \beta_9^3, \beta_9^3\beta_i, \beta_9^2, \beta_9^2\beta_j, \beta_9^2\beta_k\beta_l, \beta_9, \beta_9\beta_j, \beta_9\beta_k\beta_l, 1, \beta_j, \beta_k\beta_l,$$

where $i, j, k, l = 10, 11, 12$. Notice we don't need the term $\beta_9^3\beta_i\beta_j$ in this list. If we continue the iterations as above and whenever necessary we replace β_9^4 by the above expression but without reduction on the parts of various products of $\beta_{10}, \beta_{11}, \beta_{12}$, then we get β_9^{n+3} as a linear combination of monomials of the form

$$\beta_9^l\beta_{10}^{l_1}\beta_{11}^{l_2}\beta_{12}^{l_3},$$

for $0 \le l \le 3$ and $0 \le l_1 + l_2 + l_3 \le 2n$. By the reduction at the beginning of this proof, $\beta_{10}^{l_1}\beta_{11}^{l_2}\beta_{12}^{l_3}$ can be reduced to a combination of elements in the list of (L_1) with the degree of β_9 not bigger than $[\frac{l_1+l_2+l_3-1}{2}] \le [\frac{2n-1}{2}] = n-1$, so β_9^{n+3} is reduced to a linear combination of monomials of the form

$$\beta_9^l\beta_{10}^{l_1}\beta_{11}^{l_2}\beta_{12}^{l_3},$$

for $0 \le l \le n+2$ and $0 \le l_1 + l_2 + l_3 \le 2$.

Now we suppose powers of β_9 of degrees lower than $n+3$ can be represented as linear combinations of elements

$$(L_3) \qquad \beta_9^3, \beta_9^3\beta_i, \beta_9^3\beta_i\beta_j, \beta_9^2, \beta_9^2\beta_j, \beta_9^2\beta_k\beta_l, \beta_9, \beta_9\beta_j, \beta_9\beta_k\beta_l, 1, \beta_j, \beta_k\beta_l,$$

where $i, j, k, l = 10, 11, 12$. Then to show β_9^{n+3} can be represented by these elements, since β_9^4 can be expressed by elements in (L_2) hence in (L_3), we are left with two terms $\beta_9^4\beta_i, \beta_9^4\beta_i\beta_j$ to consider. In turn, if we look back at the list (L_2), this is reduced to $\beta_9^3\beta_i, \beta_9^3\beta_j\beta_k, \beta_9^3\beta_j\beta_k\beta_l$ among other terms already in (L_3). The first two terms are good in (L_3). Since $\beta_j\beta_k\beta_l$ can be expressed by elements

$$\beta_9, \beta_9\beta_i, 1, \beta_j, \beta_j\beta_k,$$

where $i, j, k = 10, 11, 12$, $\beta_9^3\beta_j\beta_k\beta_l$ is reduced to terms $\beta_9^4, \beta_9^4\beta_i$ among others, which are good as reasoned above. All together by induction, β_9^{n+3} can be written as a linear combination of elements in (L_3). Combined with the list (L_1), the quotient ring is generated by elements

$$\beta_9^l\beta_{10}^{l_1}\beta_{11}^{l_2}\beta_{12}^{l_3},$$

where $0 \leq l \leq 3$ and $0 \leq l_1 + l_2 + l_3 \leq 4$. $\square$

Summarizing these results, we conclude with the following

Theorem 5.4.6. *The quantum cohomology ring $QH^*(F^{[2]})$ is isomorphic to the quotient ring $\mathbb{Q}[\beta_9, \beta_{10}, \beta_{11}, \beta_{12}, q_2, q_3][[q_1]]/(f_1', f_2', \cdots, f_9')$.*

For Hirzebruch surfaces F_a, it is known that Hilbert schemes $F_a^{[2]}$ are symplectically isomorphic according to whether a is even or odd. Since F_0 is the same as $\mathbb{P}^1 \times \mathbb{P}^1$, together with the result of Pontoni[16], the quantum cohomology of the Hilbert schemes of two points for all Hirzebruch surfaces are determined.

References

[1] Batyrev,V.V. Quantum cohomology rings of toric varieties, Asterisque 218 (1991), 9-34.

[2] Behrend, K. Gromov-Witten invariants in algebraic geometry. Invent. Math. 127 (1997), no. 3, 601-617.

[3] Behrend, K.; Fantechi, B. The intrinsic normal cone. Invent. Math. 128 (1997), no. 1, 45-88.

[4] Bertram, A., Daskalopoulos,G., Wenthworth,R. Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians, J.AMS. 9(1996), no.2, 529-571.

[5] Ciocan-Fontanine, I. Quantum cohomology of flag varieties, IMRN(1995), no.6, 263-277.

[6] Edidin, Dan; Li,Wei-Ping; Qin, Zhenbo, Gromov-Witten invariants of the Hilbert scheme of 3-points on $\mathbb{P}^2$, Asian J. Math. 7 (2003), no. 4, 551-574.

[7] Cox, David A.; Katz, Sheldon, Mirror symmetry and algebraic geometry. Mathematical Surveys and Monographs, 68. American Mathematical Society, Providence, RI, 1999

[8] Fulton, William, Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1984. xi+470 pp.

[9] Fulton, W.; Pandharipande, R. Notes on stable maps and quantum cohomology. Algebraic geometry—Santa Cruz 1995, 45-96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[10] Graber, T.; Pandharipande, R. Localization of virtual classes. Invent. Math. 135 (1999), no. 2, 487-518.

[11] Hartshorne, R. Algebraic Geometry, Springer Verlag, New York, 1977,

[12] Illusie, Luc, Complexe Cotangent et Deformations I, Lecture Notes in Mathematics 239, Springer-Verlag, 1971,

[13] Li, Wei-Ping, Qin, Zhenbo, On 1-point Gromov-Witten invariants of the Hilbert schemes of points on surfaces. Turkish J. Math. 26 (2002), 53-68.

[14] Matsumura, H. Commutative ring theory, Cambridge University Press, New York, 1986.

[15] Okounkov, A., Pandharipande, R. Quantum cohomology of the Hilbert scheme of points of the plane, Invent. Math, 179(2010), 523-557.

[16] Pontoni, D. Quantum Cohomology of $Hilb^2(P^1 \times P^1)$ and Enumerative Applications, Trans. AMS, Vol.359, No.11, Nov. 2007, 5419-5448

[17] Ruan, Yongbin, Topological sigma model and Donaldson-type invariants in Gromov theory. Duke Math. J. 83 (1996), no. 2, 461-500,

[18] Ruan, Yongbin; Tian, Gang, A mathematical theory of quantum cohomology. J. Differential Geom. 42 (1995), no. 2, 259-367.

[19] Siebert, B., Tian, G. On Quantum Cohomology Rings of Fano Manifolds and a Formula of Vafa and Intriligator, The Asian J. of Math.1(1997), no.4, 679-695.

[20] Spielberg, Holger, A formula for the Gromov-Witten invariants of toric varieties. Thesis, Universite Louis Pasteur (Strasbourg I), Strasbourg, 1999.